In northern forests, where the sun is relatively low in the sky, trees tend to be tall and narrow to maximize their exposure to the available light. In contrast, in equatorial regions trees tend to have broad flat tops because the sun is mostly overhead during the day. Optimization is an important characteristic of natural systems, as well as a key factor in decision-making processes in applied problem solving. In this chapter, we will use the derivative to analyze functions and find optimal solutions in a variety of applied settings.

4 APPLICATIONS OF THE DERIVATIVE

This chapter puts the derivative to work. The first and second derivatives are used to analyze functions and their graphs and to solve optimization problems (finding minimum and maximum values of a function). Newton's Method in Section 4.8 employs the derivative to approximate solutions of equations.

4.1 Linear Approximation and Applications

In this section, we introduce the process of linear approximation that uses the tangent line to the graph of a function f at $x = a$ to approximate $f(x)$ for x near a. These approximation methods are desirable because linear functions are usually easier to use and compute with than nonlinear ones. We introduce a few different formulas involving linear approximation. There are different settings and situations where each is useful. Keep in mind that they all come from the same basic idea that the tangent line approximates the function close to the point of tangency (Figure 1).

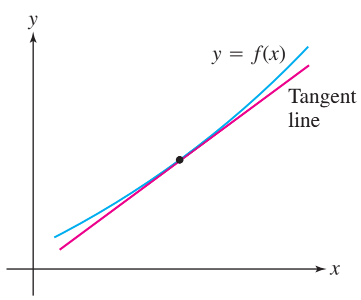


FIGURE 1 The tangent line approximates the graph of f near the point of tangency.

Linear Approximation

In some situations, we are interested in the effect of a small change. For example,

- How does a small change in angle affect the distance of a basketball shot? (Exercise 47)
- How are revenues at the box office affected by a small change in ticket prices? (Exercise 37)
- The cube root of 27 is 3. How much greater is the cube root of 27.2? (Exercise 7)

In each case, we have a function f and we're interested in the change

$$\boxed{\Delta f = f(a + \Delta x) - f(a)}$$

where Δx is small. The **Linear Approximation** uses the slope of the tangent line (i.e., the derivative) to estimate Δf without computing it exactly. By definition, the derivative is the limit

$$f'(a) = \lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$$

So when Δx is small, we have $\Delta f/\Delta x \approx f'(a)$, and thus,

$$\Delta f \approx f'(a)\Delta x$$

REMINDER The notation ≈ means "approximately equal to." The accuracy of the Linear Approximation is discussed at the end of this section.

Linear Approximation of Δf If f is differentiable at $x = a$ and Δx is small, then

$$\boxed{\Delta f \approx f'(a)\Delta x} \qquad \boxed{1}$$

It is important to understand the different roles played by Δf and $f'(a)\Delta x$. The quantity of interest is the *actual change* Δf. We estimate it by $f'(a)\,\Delta x$, the change on the tangent line with slope $f'(a)$. The Linear Approximation tells us that up to a small error, Δf is approximately equal to $f'(a)\Delta x$ when Δx is small.

GRAPHICAL INSIGHT As we indicated, the Linear Approximation is an approximation using a tangent line. In fact, it is sometimes called the **tangent line approximation**. Observe in Figure 2 that Δf is the vertical change in the graph from $x = a$ to $x = a + \Delta x$. For a line, the vertical change is equal to the slope times the horizontal change Δx, and since the tangent line has slope $f'(a)$, its vertical change is $f'(a)\Delta x$. So the Linear Approximation approximates Δf by the vertical change in the tangent line. When Δx is small, the two quantities are nearly equal.

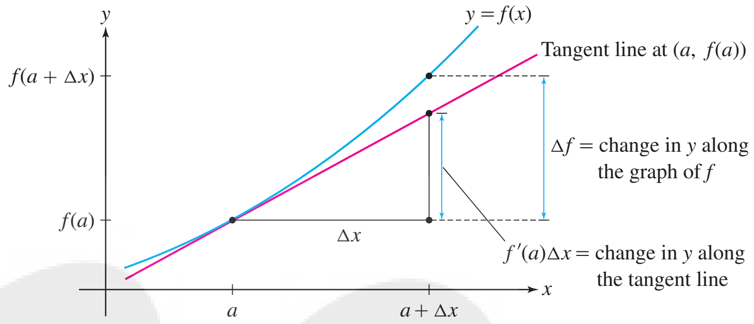


FIGURE 2 Graphical meaning of the Linear Approximation $\Delta f \approx f'(a)\Delta x$.

EXAMPLE 1 Use the Linear Approximation to estimate the change in $f(x) = 1/x$ as x goes from 10 to 10.2; that is, to estimate $\frac{1}{10.2} - \frac{1}{10}$. How accurate is the estimate?

Solution We apply the Linear Approximation to $f(x) = \frac{1}{x}$ with $a = 10$ and $\Delta x = 0.2$: We have $f'(x) = -x^{-2}$ and $f'(10) = -0.01$, so Δf is approximated by

$$\Delta f \approx f'(10)\Delta x = (-0.01)(0.2) = -0.002$$

Since $\Delta f = \frac{1}{10.2} - \frac{1}{10}$ we have the approximation

$$\frac{1}{10.2} - \frac{1}{10} \approx -0.002$$

The error in the Linear Approximation is the quantity

$$\text{error} = \left|\Delta f - f'(a)\Delta x\right|$$

A calculator gives the value $\frac{1}{10.2} - \frac{1}{10} \approx -0.00196$, and thus, our error is less than 10^{-4}:

$$\text{error} \approx \left|-0.00196 - (-0.002)\right| = 0.00004 < 10^{-4}$$ ■

EXAMPLE 2 Approximate how much greater $\sqrt[3]{8.1}$ is than $\sqrt[3]{8} = 2$, and then use the result to approximate $\sqrt[3]{8.1}$.

Solution We are interested in $\sqrt[3]{8.1} - \sqrt[3]{8}$, so we apply the Linear Approximation to $f(x) = x^{1/3}$ with $a = 8$ and $\Delta x = 0.1$. We have

$$f'(x) = \frac{1}{3}x^{-2/3} \qquad \text{and} \qquad f'(8) = \left(\frac{1}{3}\right)8^{-2/3} = \left(\frac{1}{3}\right)\left(\frac{1}{4}\right) = \frac{1}{12}$$

Therefore, $\Delta f \approx f'(8)\Delta x = \frac{1}{12}(0.1) \approx 0.0083$, and since

$$\Delta f = f(a + \Delta x) - f(a) = \sqrt[3]{8 + 0.1} - \sqrt[3]{8} = \sqrt[3]{8.1} - 2$$

we have the approximation

$$\sqrt[3]{8.1} - 2 \approx 0.0083$$

Thus, $\sqrt[3]{8.1}$ is greater than $\sqrt[3]{8}$ by approximately 0.0083. It follows that

$$\sqrt[3]{8.1} \approx 2 + 0.0083 = 2.0083$$ ■

Suppose that we measure the *diameter* D of a circle and use this result to compute the *area* of the circle. If our measurement of D is inexact, the area computation will also be inexact. What is the effect of the measurement error on the resulting area computation? This can be estimated using the Linear Approximation, as in the next example.

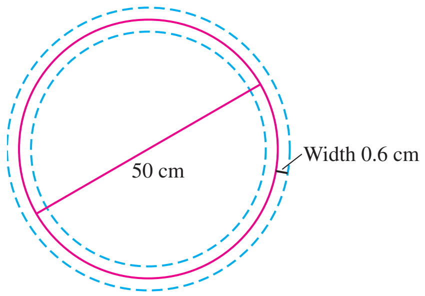


FIGURE 3 The border of the actual pizza lies between the dashed circles.

In this example, we interpret ΔA as the possible error in the computation of $A(D)$. This should not be confused with the error in the Linear Approximation. This latter error refers to the accuracy in using $A'(D)\,\Delta D$ to approximate ΔA.

EXAMPLE 3 Effect of an Inexact Measurement The Cheezy Pizza Parlor claims that its pizzas are circular with diameter 50 cm (Figure 3).

(a) What is the area of the pizza?

(b) Estimate the quantity of pizza lost or gained if the diameter is off by at most 1.2 cm.

Solution First, we need a formula for the area A of a circle in terms of its diameter D. Since the radius is $r = D/2$, the area is

$$A(D) = \pi r^2 = \pi \left(\frac{D}{2}\right)^2 = \frac{\pi}{4} D^2$$

(a) If $D = 50$ cm, then the pizza has area $A(50) = \left(\frac{\pi}{4}\right)(50)^2 \approx 1963.5 \text{ cm}^2$.

(b) If the actual diameter is equal to $50 + \Delta D$, then the loss or gain in pizza area is $A(50 + \Delta D) - A(50) = \Delta A$. We apply Linear Approximation to $A(D)$ with $D = 50$ and $\Delta D = \pm 1.2$. Observe that $A'(D) = \frac{\pi}{2} D$ and $A'(50) = 25\pi \approx 78.5$ cm, so the Linear Approximation yields

$$\Delta A \approx A'(D)\Delta D \approx (78.5)\,\Delta D$$

Because ΔD is at most ± 1.2 cm, the loss or gain in pizza is no more than around

$$\Delta A \approx \pm(78.5)(1.2) \approx \pm 94.2 \text{ cm}^2$$

This is a loss or gain of approximately 4.8% of the area of 1963.5 cm^2. ■

Linearization

To approximate the function f itself rather than the change Δf, we use the **linearization** $L(x)$ **centered at** $x = a$, defined by

$$L(x) = f'(a)(x - a) + f(a)$$

Notice that $y = L(x)$ is the equation of the tangent line at $x = a$. For values of x close to a, $L(x)$ provides a good approximation to $f(x)$ (Figure 4).

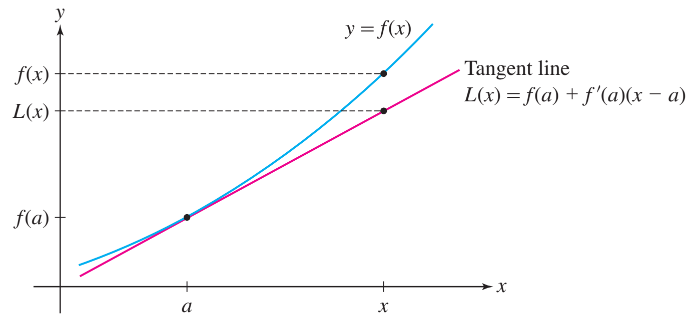


FIGURE 4 For x near a, the function value $f(x)$ is approximated by the tangent line value $L(x)$.

Approximating f by Its Linearization If f is differentiable at a and x is close to a, then $f(x) \approx L(x)$, so

$$\boxed{f(x) \approx f(a) + f'(a)(x - a)}$$

Note that, by rearranging the terms in linearization formula, we obtain the Linear Approximation formula, $\Delta f \approx f'(a)\Delta x$, that we introduced previously. Specifically, with $\Delta x = x - a$ and $\Delta f = f(x) - f(a)$, we have

$$f(x) \approx f(a) + f'(a)(x-a)$$
$$f(x) - f(a) \approx f'(a)\,\Delta x \qquad (\text{since } \Delta x = x - a)$$
$$\Delta f \approx f'(a)\Delta x$$

EXAMPLE 4 Determine the approximation formula for $f(x) = \sqrt{x}e^{x-1}$ resulting from the linearization at $a = 1$.

Solution The linearization at $a = 1$ is the approximation formula that is given by $f(x) \approx f(1) + f'(1)(x-1)$. Note that $f(1) = \sqrt{1}e^{1-1} = 1$. Then, using the Product Rule to compute the derivative, we obtain

$$f'(x) = \frac{1}{2}x^{-1/2}e^{x-1} + x^{1/2}e^{x-1} = \left(\frac{1}{2}x^{-1/2} + x^{1/2}\right)e^{x-1}$$

and therefore, $f'(1) = \left(\frac{1}{2} + 1\right)e^0 = \frac{3}{2}$. Thus,

$$f(1) + f'(1)(x-1) = 1 + \frac{3}{2}(x-1) = \frac{3}{2}x - \frac{1}{2}$$

This yields the approximation formula, valid for x close to 1:

$$\sqrt{x}e^{x-1} \approx \frac{3}{2}x - \frac{1}{2}$$

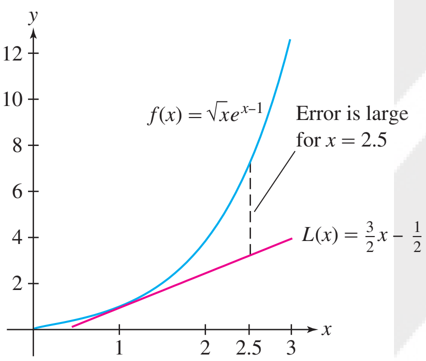


FIGURE 5 Graph of $f(x) = \sqrt{x}e^{x-1}$ and its linearization at $a = 1$.

The following table compares values of the linearization to values obtained from a calculator for the function $f(x) = \sqrt{x}e^{x-1}$ in the previous example. Note that the error is large for $x = 2.5$, as expected, because 2.5 is not close to the center of the linearization $a = 1$ (Figure 5).

x	$\sqrt{x}e^{x-1}$	**Linearization** $\frac{3}{2}x - \frac{1}{2}$	**Calculator**	**Error**
1.1	$\sqrt{1.1}e^{0.1}$	$\frac{3}{2}(1.1) - \frac{1}{2} = 1.15$	1.15911	10^{-2}
0.999	$\sqrt{0.999}e^{-0.001}$	$\frac{3}{2}(0.999) - \frac{1}{2} = 0.9985$	0.998501	10^{-6}
2.5	$\sqrt{2.5}e^{1.5}$	$\frac{3}{2}(2.5) - \frac{1}{2} = 3.25$	7.086	3.84

In the next example, we compute the **percentage error**, which is often more important than the error itself because it gives us a measure of how large the error is in relation to the actual value. An error of 0.1 is more significant when the actual value is 3 than when the actual value is 333. By definition,

$$\boxed{\text{percentage error} = \left|\frac{\text{error}}{\text{actual value}}\right| \times 100\%}$$

EXAMPLE 5 Estimate $\tan\left(\frac{\pi}{4} + 0.02\right)$ and compute the percentage error.

Solution We use the linearization of $f(x) = \tan x$ at $a = \frac{\pi}{4}$ for our approximation. So we need to calculate the terms in $f(\pi/4) + f'(\pi/4)(x - \pi/4)$:

$$f\left(\frac{\pi}{4}\right) = \tan\left(\frac{\pi}{4}\right) = 1, \qquad f'\left(\frac{\pi}{4}\right) = \sec^2\left(\frac{\pi}{4}\right) = \left(\sqrt{2}\right)^2 = 2$$
$$f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right) = 1 + 2\left(x - \frac{\pi}{4}\right)$$

So, for x near $\pi/4$, we have the approximation formula

$$\tan(x) \approx 1 + 2\left(x - \frac{\pi}{4}\right)$$

At $x = \frac{\pi}{4} + 0.02$, this approximation yields the estimate

$$\tan\left(\frac{\pi}{4} + 0.02\right) \approx 1 + 2\left(\frac{\pi}{4} + 0.02 - \frac{\pi}{4}\right) = 1.04$$

A calculator gives $\tan\left(\frac{\pi}{4} + 0.02\right) \approx 1.0408$, so

$$\text{percentage error} \approx \left|\frac{1.0408 - 1.04}{1.0408}\right| \times 100 \approx 0.08\%$$

■

Differential Form of Linear Approximation

Another way of expressing the Linear Approximation is via the **differentials** dx and dy that represent the change in x and y, respectively, on the tangent line to $f(x)$ at $x = a$. Since these differentials represent change on the tangent line, we have

$$\boxed{dy = f'(a)dx} \qquad \boxed{2}$$

As before, we let Δy represent the change in y on the graph of f. It follows—as in the previous approximations in this section—that with a small change in x, the change in y on the graph is approximately the change in y on the tangent line (Figure 6). Thus, $\Delta y \approx dy$, yielding the following:

As we mentioned before, in the Leibniz notation for the derivative, $\frac{dy}{dx}$ does not represent a fraction. It is via differentials, though, that the relationship $dy = \frac{dy}{dx}dx$ is made mathematically meaningful. We will find relationships like this to be very useful when simplifying computations involving integrals in subsequent chapters.

Differential Form of Linear Approximation If f is differentiable at a and dx is small, then

$$\boxed{\Delta y \approx dy = f'(a)dx} \qquad \boxed{3}$$

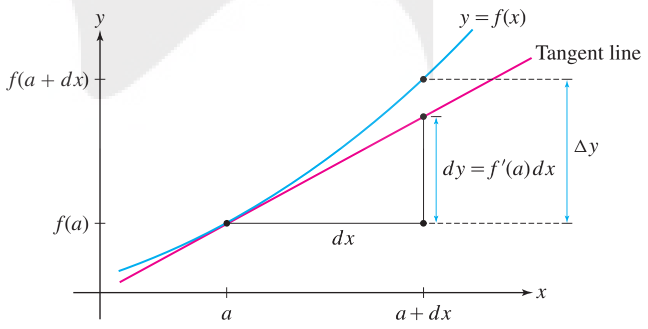


FIGURE 6 The approximation $\Delta y \approx dy = f'(a)dx$.

CONCEPTUAL INSIGHT At the start of the section, we observed that all of the approximation relationships presented here are based on the idea that the tangent line is a good approximation to the graph of the function near the point of tangency. The Linear Approximation, the linearization, and the Differential Form of Linear Approximation are illustrated in Figures 2, 4, and 6, respectively. Note that these figures all depict the graph of f and the tangent line at $x = a$. From figure to figure, various features are described or labeled differently in order to illustrate the important aspects of each approximation relationship.

You might wonder why we bother with the Differential Form of Linear Approximation. At this point, it just appears to be another way of expressing a relationship that we already had a perfectly good way of expressing. The intent here is to provide an initial glimpse into a tool, the differential, that is employed often by mathematicians, scientists, and engineers to express or approximate a small change involving related variables. A differential corresponds to the change on the tangent line (as we see here), on the tangent plane (see Section 14.4), or in the tangent space (in higher dimensions). Differentials provide a straightforward linear means for approximating and working with complicated relationships.

EXAMPLE 6 Thermal Expansion Changes in temperature can have subtle effects on physical properties of objects that we might think normally are constant. A thin metal cable has length $L = 12$ cm when the temperature is $T = 21°$C. Estimate the change in length when T rises to 24°C, assuming that

$$\frac{dL}{dT} = kL \qquad \boxed{4}$$

where $k = 1.7 \times 10^{-5}{}^\circ\text{C}^{-1}$ (k is called the coefficient of thermal expansion).

Solution How does the Linear Approximation apply here? We will use the differential dL to estimate the actual change in length ΔL when T increases from 21°C to 24°C—that is, when $dT = \Delta T = 3$°C. By Eq. (2), the differential dL is

$$dL = \left(\frac{dL}{dT}\right) dT$$

By Eq. (4), since $L = 12$,

$$\left.\frac{dL}{dT}\right|_{L=12} = kL = (1.7 \times 10^{-5})(12) \approx 2 \times 10^{-4} \text{ cm/}^\circ\text{C}$$

Therefore, with $dT = 3$, we have

$$dL = \left(\frac{dL}{dT}\right) dT \approx (2 \times 10^{-4})(3) = 6 \times 10^{-4} \text{ cm}$$

Thus, $\Delta L \approx dL$ tells us that when the temperature increases from 21°C to 24°C, we can expect the cable to lengthen by approximately 0.0006 cm. ■

The Size of the Error

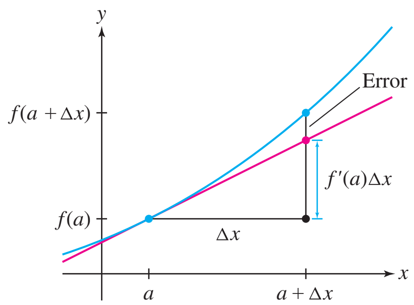


FIGURE 7 Graphical interpretation of the error in the Linear Approximation.

The examples in this section may have convinced you that the Linear Approximation yields a good approximation to Δf when Δx is small, but if we want to rely on the Linear Approximation, we need to know more about the size of the error:

$$E = \text{error} = \left|\Delta f - f'(a)\Delta x\right|$$

Graphically the error E is the vertical gap between the graph of f and the tangent line (Figure 7). In Section 10.7, we will prove the following **Error Bound**:

$$\boxed{E \le \frac{1}{2} K (\Delta x)^2} \qquad \boxed{5}$$

where K is the maximum value of $|f''(x)|$ on the interval from a to $a + \Delta x$.

The Error Bound tells us two important things. First, it says that the error is small when the second derivative (and hence K) is small. This makes sense, because $f''(x)$ measures how quickly the tangent lines change direction. When $|f''(x)|$ is smaller, the graph is flatter and the Linear Approximation is more accurate over a larger interval around $x = a$ (compare the graphs in Figure 8).

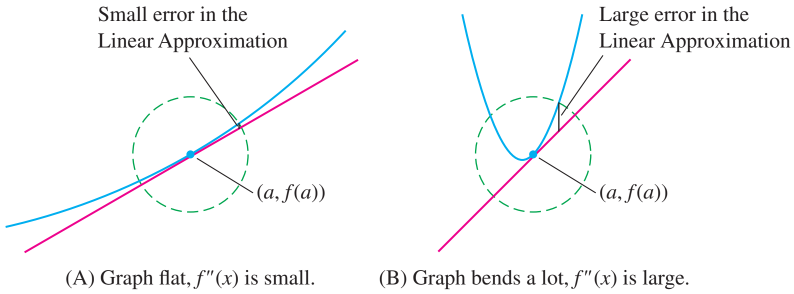


FIGURE 8 The accuracy of the Linear Approximation depends on how much the curve bends.

Second, the Error Bound tells us that the error is of *order 2* in Δx, meaning that E is no larger than a constant times $(\Delta x)^2$. So if Δx is small, say, $\Delta x = 10^{-n}$, then E has a substantially smaller order of magnitude, since $(\Delta x)^2 = 10^{-2n}$. In particular, $E/\Delta x$ tends to zero (because $E/\Delta x < K\,\Delta x$), so the Error Bound tells us that the graph becomes nearly indistinguishable from its tangent line as we zoom in on the graph around $x = a$. This is a precise version of the "local linearity" property discussed in Section 3.2.

4.1 SUMMARY

The approximation formulas in this section are all based on the idea that the tangent line to the graph of a function f at $x = a$ can be used to approximate $f(x)$ for x near a.

- Let $\Delta f = f(a + \Delta x) - f(a)$. The *Linear Approximation* is the estimate

$$\boxed{\Delta f \approx f'(a)\Delta x \quad (\text{for } \Delta x \text{ small})}$$

- The *linearization of* $f(x)$ *centered at* $x = a$ is the function for the tangent line

$$\boxed{L(x) = f(a) + f'(a)(x - a)}$$

- The approximation based on linearization is

$$\boxed{f(x) \approx f(a) + f'(a)(x - a) \qquad (\text{for } x \text{ close to } a)}$$

- Differential notation: $dx = \Delta x$ is the change in x, $dy = f'(a)dx$ is the change on the tangent line, and $\Delta y = f(a + \Delta x) - f(a)$ is the change in f. In this notation, the *Differential Form of Linear Approximation* is

$$\boxed{\Delta y \approx dy = f'(a)dx \qquad (\text{for } dx \text{ small})}$$

- The error in the Linear Approximation is the quantity

$$\text{error} = \left|\Delta f - f'(a)\Delta x\right|$$

The percentage error is often more significant because it is a measure of how large the error is in relation to the actual value:

$$\text{percentage error} = \left|\frac{\text{error}}{\text{actual value}}\right| \times 100\%$$

- The error E in the Linear Approximation is bounded as follows:

$$E \le \frac{1}{2}K\,(\Delta x)^2$$

where K is the maximum value of $|f''(x)|$ on the interval from a to $a + \Delta x$.

4.1 EXERCISES

Preliminary Questions

1. True or False? The Linear Approximation says that the vertical change in the graph is approximately equal to the vertical change in the tangent line.

2. Estimate $g(1.2) - g(1)$ if $g'(1) = 4$.

3. Estimate $f(2.1)$ if $f(2) = 1$ and $f'(2) = 3$.

4. Complete the following sentence: The Linear Approximation shows that up to a small error, the change in output Δf is directly proportional to ______________ .

Exercises

In Exercises 1–6, use Eq. (1) to estimate $\Delta f = f(3.02) - f(3)$.

1. $f(x) = x^2$

2. $f(x) = x^4$

3. $f(x) = x^{-1}$

4. $f(x) = \dfrac{1}{x+1}$

5. $f(x) = \sqrt{x+6}$

6. $f(x) = \tan \dfrac{\pi x}{3}$

7. The cube root of 27 is 3. How much larger is the cube root of 27.2? Estimate using the Linear Approximation.

8. The cube root of 64 is 4. How much smaller is the cube root of 63.6? Estimate using the Linear Approximation.

In Exercises 9–12, use Eq. (1) to estimate Δf. Use a calculator to compute both the error and the percentage error.

9. $f(x) = \sqrt{1+x}, \quad a = 3, \quad \Delta x = 0.2$

10. $f(x) = 2x^2 - x, \quad a = 5, \quad \Delta x = -0.4$

11. $f(x) = \dfrac{1}{1+x^2}, \quad a = 3, \quad \Delta x = 0.5$

12. $f(x) = \ln(x^2+1), \quad a = 1, \quad \Delta x = 0.1$

In Exercises 13–20, using Linear Approximation, estimate Δf for a change in x from $x = a$ to $x = b$. Use the estimate to approximate $f(b)$, and find the error using a calculator.

13. $f(x)=\sqrt{x}, a=25, b=26$

14. $f(x)=x^{1/4}, a=16, b=16.5$

15. $f(x)=\frac{1}{\sqrt{x}}, a=100, b=101$

16. $f(x)=\frac{1}{\sqrt{x}}, a=100, b=98$

17. $f(x)=x^{1/3}, a=8, b=9$

18. $f(x)=\tan^{-1} x, a=1, b=1.05$

19. $f(x)=e^x, a=0, b=-0.1$

20. $f(x)=\ln x, a=1, b=0.97$

In Exercises 21–28, find the linearization at $x = a$ and then use it to approximate $f(b)$.

21. $f(x) = x^4, \quad a = 1, \quad b = 0.96$

22. $f(x) = \dfrac{1}{x}, \quad a = 2, \quad b = 2.02$

23. $f(x) = \sin^2 x, \quad a = \frac{\pi}{4}, \quad b = \frac{1.1\pi}{4}$

24. $f(x) = \dfrac{x^2}{x-3}, \quad a = 4, \quad b = 4.1$

25. $f(x) = (1+x)^{-1/2}, \quad a = 0, \quad b = 0.08$

26. $f(x) = (1+x)^{-1/2}, \quad a = 3, \quad b = 2.88$

27. $f(x) = e^{\sqrt{x}}, \quad a = 1, \quad b = 0.85$

28. $f(x) = e^x \ln x, \quad a = 1, \quad b = 1.02$

In Exercises 29–32, estimate Δy using differentials [Eq. (3)].

29. $y = \cos x, \quad a = \frac{\pi}{6}, \quad dx = 0.014$

30. $y = \tan^2 x, \quad a = \frac{\pi}{4}, \quad dx = -0.02$

31. $y = \dfrac{10 - x^2}{2 + x^2}, \quad a = 1, \quad dx = 0.01$

32. $y = x^{1/3}e^{x-1}, \quad a = 1, \quad dx = 0.1$

33. Estimate $f(4.03)$ for $f(x)$ as in Figure 9.

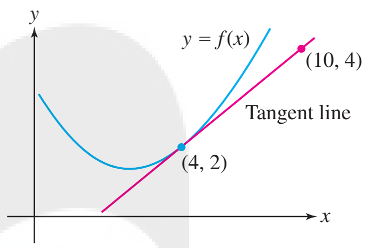


FIGURE 9

34. At a certain moment, an object in linear motion has velocity 100 m/s. Estimate the distance traveled over the next quarter-second, and explain how this is an application of the Linear Approximation.

35. Which is larger: $\sqrt{2.1} - \sqrt{2}$ or $\sqrt{9.1} - \sqrt{9}$? Explain using the Linear Approximation.

36. Estimate $\sin 61^\circ - \sin 60^\circ$ using the Linear Approximation. *Hint:* Express $\Delta\theta$ in radians.

37. Box office revenue at a cinema in Paris is $R(p) = 3600p - 10p^3$ euros per showing when the ticket price is p euros. Calculate $R(p)$ for $p = 9$ and use the Linear Approximation to estimate ΔR if p is raised or lowered by 0.5 euro.

38. The *stopping distance* for an automobile is $F(s) = 1.1s + 0.054s^2$ ft, where s is the speed in mph. Use the Linear Approximation to estimate the change in stopping distance per additional mph when $s = 35$ and when $s = 55$.

39. A thin silver wire has length $L = 18$ cm when the temperature is $T = 30^\circ$C. Estimate ΔL when T decreases to 25°C if the coefficient of thermal expansion is $k = 1.9 \times 10^{-5}\,^\circ\text{C}^{-1}$ (see Example 6).

40. At a certain moment, the temperature in a snake cage satisfies $dT/dt = 0.008^\circ$C/s. Estimate the rise in temperature over the next 10 s.

41. The atmospheric pressure at altitude h (kilometers) for $11 \le h \le 25$ is approximately

$$P(h) = 128e^{-0.157h} \text{ kilopascals}$$

(a) Estimate ΔP at $h = 20$ when $\Delta h = 0.5$.

(b) Compute the actual change, and compute the percentage error in the Linear Approximation.

42. The resistance R of a copper wire at temperature $T = 20°$C is $R = 15\ \Omega$. Estimate the resistance at $T = 22°$C, assuming that $dR/dT\big|_{T=20} = 0.06\ \Omega/°$C.

43. Newton's Law of Gravitation shows that if a person weighs w pounds on the surface of the earth, then his or her weight at distance x from the center of the earth is

$$W(x) = \frac{wR^2}{x^2} \qquad (\text{for } x \geq R)$$

where $R = 3960$ miles is the radius of the earth (Figure 10).

(a) Show that the weight lost at altitude h miles above the earth's surface is approximately $\Delta W \approx -(0.0005w)h$. *Hint:* Use the Linear Approximation with $dx = h$.

(b) Estimate the weight lost by a 200-lb football player flying in a jet at an altitude of 7 miles.

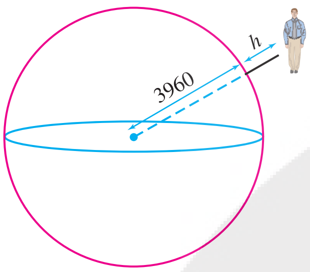


FIGURE 10 The distance to the center of the earth is $3960 + h$ miles.

44. Using Exercise 43(a), estimate the altitude at which a 130-lb pilot would weigh 129.5 lb.

45. A stone tossed vertically into the air with initial velocity v cm/s reaches a maximum height of $h = v^2/1960$ cm.

(a) Estimate Δh if $v = 700$ cm/s and $\Delta v = 1$ cm/s.

(b) Estimate Δh if $v = 1000$ cm/s and $\Delta v = 1$ cm/s.

(c) In general, does a 1-cm/s increase in v lead to a greater change in h at low or high initial velocities? Explain.

46. The side s of a square carpet is measured at 6 m. Estimate the maximum error in the area A of the carpet if s is accurate to within 2 cm.

In Exercises 47 and 48, use the following fact derived from Newton's Laws: An object released at an angle θ with initial velocity v ft/s travels a horizontal distance

$$s = \frac{1}{32}v^2 \sin 2\theta \text{ ft (Figure 11)}$$

47. A player located 18.1 ft from the basket launches a successful jump shot from a height of 10 ft (level with the rim of the basket), at an angle $\theta = 34°$ and initial velocity $v = 25$ ft/s.

(a) Show that $\Delta s \approx 0.255\Delta\theta$ ft for a small change of $\Delta\theta$.

(b) Is it likely that the shot would have been successful if the angle had been off by 2°?

(c) Estimate Δs if $\theta = 34°$, $v = 25$ ft/s, and $\Delta v = 2$.

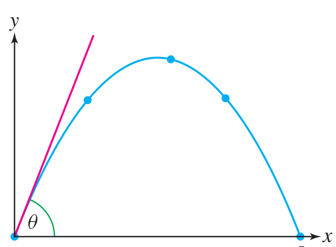


FIGURE 11 Trajectory of an object released at an angle θ.

48. A golfer hits a golf ball at an angle of $\theta = 23°$ with initial velocity $v = 120$ ft/s.

(a) Estimate Δs if the ball is hit at the same velocity but the angle is increased by 3°.

(b) Estimate Δs if the ball is hit at the same angle but the velocity is increased by 3 ft/s.

49. The radius of a spherical ball is measured at $r = 25$ cm. Estimate the maximum error in the volume and surface area if r is accurate to within 0.5 cm.

50. The dosage D of diphenhydramine for a dog of body mass w kg is $D = 4.7w^{2/3}$ mg. Estimate the maximum allowable error in w for a cocker spaniel of mass $w = 10$ kg if the percentage error in D must be less than 3%.

51. The volume (in liters) and pressure P (in atmospheres) of a certain gas satisfy $PV = 24$. A measurement yields $V = 4$ with a possible error of ± 0.3 L. Compute P and estimate the maximum error in this computation.

52. In the notation of Exercise 51, assume that a measurement yields $V = 4$. Estimate the maximum allowable error in V if P must have an error of less than 0.2 atm.

53. Approximate $f(2)$ if the linearization of $f(x)$ at $a = 2$ is $L(x) = 2x + 4$.

54. Compute the linearization of $f(x) = 3x - 4$ at $a = 0$ and $a = 2$. Prove more generally that a linear function coincides with its linearization at $x = a$ for all a.

55. Estimate $\sqrt{16.2}$ using the linearization $L(x)$ of $f(x) = \sqrt{x}$ at $a = 16$. Plot f and L on the same set of axes and from the plot indicate whether the estimate is greater than or less than the actual value.

56. GU Estimate $1/\sqrt{15}$ using a suitable linearization of $f(x) = 1/\sqrt{x}$. Plot f and L on the same set of axes and from the plot indicate whether the estimate is greater than or less than the actual value. Use a calculator to compute the percentage error.

In Exercises 57–65, approximate using linearization and use a calculator to compute the percentage error.

57. $\dfrac{1}{\sqrt{17}}$

58. $\dfrac{1}{101}$

59. $\dfrac{1}{(10.03)^2}$

60. $(17)^{1/4}$

61. $(64.1)^{1/3}$

62. $(1.2)^{5/3}$

63. $\cos^{-1}(0.52)$

64. $\ln 1.07$

65. $e^{-0.012}$

66. GU Compute the linearization $L(x)$ of $f(x) = x^2 - x^{3/2}$ at $a = 4$. Then plot $f - L$ and identify an interval I around $a = 4$ such that $|f(x) - L(x)| \leq 0.1$ for $x \in I$.

67. Show that the Linear Approximation to $f(x) = \sqrt{x}$ at $x = 9$ yields the estimate $\sqrt{9 + h} - 3 \approx \frac{1}{6}h$. Set $K = 0.01$ and show that $|f''(x)| \leq K$ for $x \geq 9$. Then verify numerically that the error E satisfies Eq. (5) for $h = 10^{-n}$, for $1 \leq n \leq 4$.

68. GU The Linear Approximation to $f(x) = \tan x$ at $x = \frac{\pi}{4}$ yields the estimate $\tan\left(\frac{\pi}{4} + h\right) - 1 \approx 2h$. Set $K = 6.2$ and show, using a plot, that $|f''(x)| \leq K$ for $x \in [\frac{\pi}{4}, \frac{\pi}{4} + 0.1]$. Then verify numerically that the error E satisfies Eq. (5) for $h = 10^{-n}$, for $1 \leq n \leq 4$.

Further Insights and Challenges

69. Compute dy/dx at the point $P = (2, 1)$ on the curve $y^3 + 3xy = 7$ and show that the linearization at P is $L(x) = -\frac{1}{3}x + \frac{5}{3}$. Use $L(x)$ to estimate the y-coordinate of the point on the curve where $x = 2.1$.

70. Apply the method of Exercise 69 to $P = (0.5, 1)$ on $y^5 + y - 2x = 1$ to estimate the y-coordinate of the point on the curve where $x = 0.55$.

71. Apply the method of Exercise 69 to $P = (-1, 2)$ on $y^4 + 7xy = 2$ to estimate the solution of $y^4 - 7.7y = 2$ near $y = 2$.

72. Show that for any real number k, $(1 + \Delta x)^k \approx 1 + k\Delta x$ for small Δx. Estimate $(1.02)^{0.7}$ and $(1.02)^{-0.3}$.

73. Let $\Delta f = f(5 + h) - f(5)$, where $f(x) = x^2$. Verify directly that $E = |\Delta f - f'(5)h|$ satisfies (5) with $K = 2$.

74. Let $\Delta f = f(1 + h) - f(1)$, where $f(x) = x^{-1}$. Show directly that $E = |\Delta f - f'(1)h|$ is equal to $h^2/(1 + h)$. Then prove that $E \le 2h^2$ if $-\frac{1}{2} \le h \le \frac{1}{2}$. *Hint:* In this case, $\frac{1}{2} \le 1 + h \le \frac{3}{2}$.

4.2 Extreme Values

In many applications, it is important to find the minimum or maximum value of a function f. For example, a physician needs to know the maximum drug concentration in a patient's bloodstream when a drug is administered. This amounts to finding the highest point on the graph of C, the concentration at time t (Figure 1).

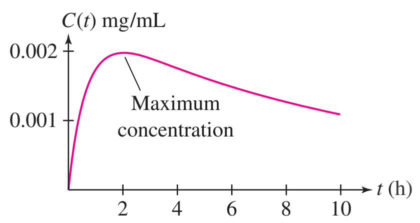


FIGURE 1 Drug concentration in bloodstream (see Exercise 78).

We refer to the maximum and minimum values (max and min for short) as **extreme values** or **extrema** (singular: extremum) and to the process of finding them as **optimization**. Sometimes, we are interested in finding the min or max for x in a particular interval I, rather than on the entire domain of f.

Often, we drop the word "absolute" and speak simply of the min or max on an interval I. When no interval is mentioned, it is understood that we refer to the extreme values on the entire domain of the function.

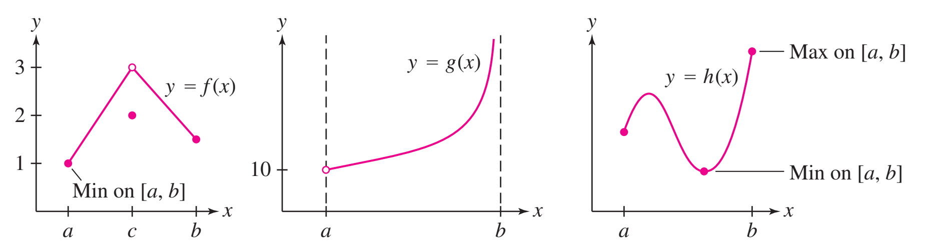
DEFINITION Extreme Values on an Interval Let f be a function on an interval I and let $a \in I$. We say that $f(a)$ is the

- **Absolute minimum** of f on I if $f(a) \le f(x)$ for all $x \in I$.
- **Absolute maximum** of f on I if $f(a) \ge f(x)$ for all $x \in I$.

Does every function have a minimum or maximum value? Clearly not, as we see by taking $f(x) = x$. Indeed, $f(x) = x$ increases without bound as $x \to \infty$ and decreases without bound as $x \to -\infty$. In fact, extreme values do not always exist even if we restrict ourselves to an interval I. Figure 2 illustrates what can go wrong if I is open or f has a discontinuity.

- **Discontinuity:** (A) shows a discontinuous function with no maximum value. The values of $f(x)$ get arbitrarily close to 3 from below, but 3 is not the maximum value because $f(x)$ never actually takes on the value 3.
- **Open interval:** In (B), $g(x)$ is defined on the *open* interval (a, b). It has no max because it tends to ∞ on the right, and it has no min because it tends to 10 on the left without ever reaching this value.

Fortunately, our next theorem guarantees that extreme values exist when the function is continuous and I is closed [Figure 2(C)].



(A) Discontinuous function with no max on $[a, b]$, and a min at $x = a$.

(B) Continuous function with no min or max on the open interval (a, b).

(C) Every continuous function on a closed interval $[a, b]$ has both a min and a max on $[a, b]$.

FIGURE 2

REMINDER *A closed, bounded interval is an interval $I = [a, b]$ (endpoints included), where a and b are both finite. Often, we drop the word "bounded" and refer to I more simply as a closed interval. An open interval (a, b) (endpoints not included) may have one or two infinite endpoints.*

THEOREM 1 Existence of Extrema on a Closed Interval A continuous function f on a closed (bounded) interval $I = [a, b]$ takes on both a minimum and a maximum value on I.

CONCEPTUAL INSIGHT Why does Theorem 1 require a closed interval? Think of the graph of a continuous function as a string. If the interval is closed, the string is pinned down at the two endpoints and cannot fly off to infinity or approach a min/max without reaching it as in Figure 2(B). Intuitively, therefore, it must have a highest and lowest point. As with the Intermediate Value Theorem in Section 2.8, a rigorous proof of Theorem 1 relies on the *completeness property* of the real numbers (see Appendix B).

Local Extrema and Critical Points

We focus now on the problem of finding extreme values. A key concept is that of a local minimum or maximum.

DEFINITION Local Extrema We say that $f(c)$ is a

- **Local minimum** occurring at $x = c$ if $f(c)$ is the minimum value of f on some open interval (in the domain of f) containing c.
- **Local maximum** occurring at $x = c$ if $f(c)$ is the maximum value of f on some open interval (in the domain of f) containing c.

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FIGURE 3 In the region surrounding Denali in Alaska, there are many local maxima, but there is one global maximum, the peak of Denali.

A local max occurs at $x = c$ if $(c, f(c))$ is the highest point on the graph within some small box [Figure 4(A)]. Thus, $f(c)$ is greater than or equal to all other *nearby* values, but it does not have to be the absolute maximum value of f (Figure 3). Local minima are similar. On the other hand, as Figure 4(B) illustrates, an absolute maximum of f on an interval $[a, b]$ need not be a local maximum of f in open intervals containing the point. In the figure, $f(a)$ is the absolute max on $[a, b]$ but is not a local max on open intervals containing a because $f(x)$ takes on greater values to the left of $x = a$.

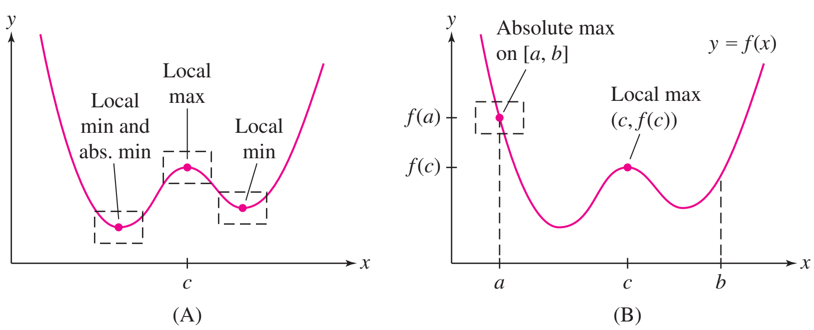
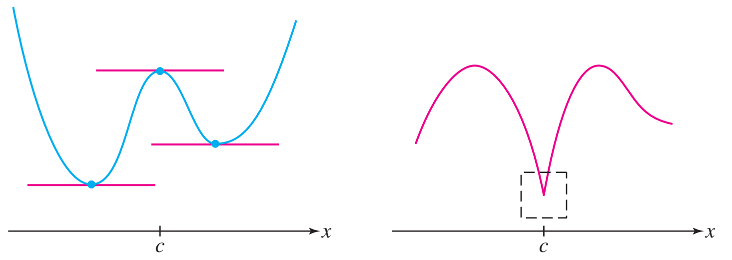


FIGURE 4

How do we find the local extrema? The crucial observation is that *the tangent line at a local min or max is horizontal* [Figure 5(A)]. In other words, if $f(c)$ is a local min or max, then $f'(c) = 0$. However, this assumes that f is differentiable. Otherwise, the tangent line may not exist, as in Figure 5(B). To take both possibilities into account, we define the notion of a critical point.



(A) Tangent line is horizontal at the local extrema.

(B) This local minimum occurs at a point where the function is not differentiable.

FIGURE 5

DEFINITION Critical Points A number c in the domain of f is called a **critical point** if *either* $f'(c) = 0$ or $f'(c)$ does not exist.

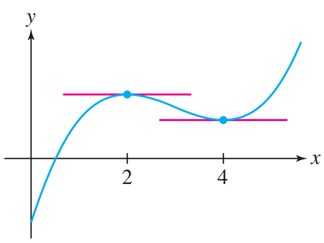


FIGURE 6 Graph of $f(x) = x^3 - 9x^2 + 24x - 10$.

EXAMPLE 1 Find the critical points of $f(x) = x^3 - 9x^2 + 24x - 10$.

Solution The function f is differentiable everywhere (Figure 6). Therefore, the critical points are the solutions of $f'(x) = 0$:

$$f'(x) = 3x^2 - 18x + 24 = 3(x^2 - 6x + 8) = 3(x - 2)(x - 4)$$

To find the critical points, we solve $3(x - 2)(x - 4) = 0$. Thus, they are $x = 2$ and $x = 4$. ■

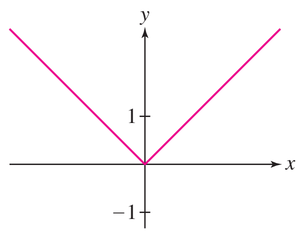


FIGURE 7 Graph of $f(x) = |x|$.

EXAMPLE 2 Nondifferentiable Function Find the critical points of $f(x) = |x|$.

Solution As we see in Figure 7, $f'(x) = -1$ for $x < 0$ and $f'(x) = 1$ for $x > 0$. Therefore, $f'(x) = 0$ has no solutions with $x \neq 0$. However, $f'(0)$ does not exist. Thus, $c = 0$ is a critical point. ■

The next theorem tells us that we can find local extrema by solving for the critical points. It is one of the most important results in calculus.

THEOREM 2 Fermat's Theorem on Local Extrema If $f(c)$ is a local min or max, then c is a critical point of f.

Proof Suppose that $f(c)$ is a local minimum (the case of a local maximum is similar). If $f'(c)$ does not exist, then c is a critical point and there is nothing more to prove. So, assume that $f'(c)$ exists. We must then prove that $f'(c) = 0$.

Because $f(c)$ is a local minimum, we have $f(c + h) \geq f(c)$ for all sufficiently small $h \neq 0$. Equivalently, $f(c + h) - f(c) \geq 0$. Now divide this inequality by h. Two possibilities occur depending on whether we are dividing by a positive value or a negative one:

$$\frac{f(c+h) - f(c)}{h} \geq 0 \qquad \text{if } h > 0 \tag{1}$$

$$\frac{f(c+h) - f(c)}{h} \leq 0 \qquad \text{if } h < 0 \tag{2}$$

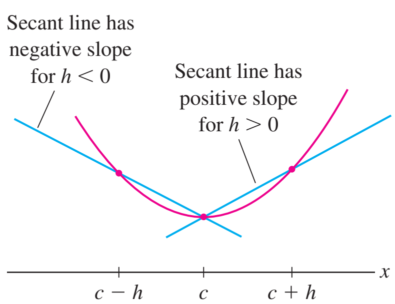


FIGURE 8

Figure 8 shows the graphical interpretation of these inequalities. Taking the one-sided limits of both sides of (1) and (2), we obtain

$$f'(c) = \lim_{h \to 0+} \frac{f(c+h) - f(c)}{h} \geq \lim_{h \to 0+} 0 = 0$$

$$f'(c) = \lim_{h \to 0-} \frac{f(c+h) - f(c)}{h} \leq \lim_{h \to 0-} 0 = 0$$

Thus, $f'(c) \geq 0$ and $f'(c) \leq 0$. The only possibility is $f'(c) = 0$ as claimed. ■

CONCEPTUAL INSIGHT Theorem 2 indicates that a local max or min must be a critical point . However, the "converse" need not be true. That is, having a critical point c does not guarantee a local min or max occurs at c. For example, $f(x) = x^3$ has derivative $f'(x) = 3x^2$ and $f'(0) = 0$. So, 0 is a critical point, but $f(0)$ is neither a local min nor max (Figure 9). The origin is a point of inflection (studied in Section 4.4), where the tangent line crosses the graph.

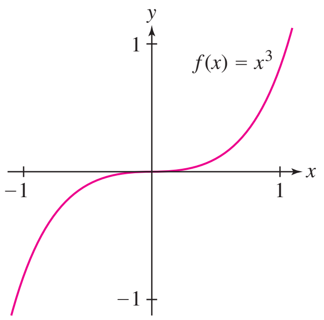


FIGURE 9 The tangent line at $(0, 0)$ is horizontal, but $f(0)$ is not a local min or max.

Optimizing on a Closed Interval

Finally, we have all the tools needed for optimizing a continuous function on a closed interval. Theorem 1 guarantees that the extreme values exist, and the next theorem tells us where to find them, namely among the critical points or endpoints of the interval.

THEOREM 3 Extreme Values on a Closed Interval Assume that f is continuous on $[a, b]$ and let $f(c)$ be the minimum or maximum value on $[a, b]$. Then c is either a critical point or one of the endpoints a or b.

Proof If c is one of the endpoints a or b, there is nothing to prove. If not, then c belongs to the open interval (a, b). In this case, $f(c)$ is also a local min or max because it is the min or max on (a, b). By Fermat's Theorem, c is a critical point. ■

EXAMPLE 3 Find the extrema of $f(x) = 2x^3 - 15x^2 + 24x + 7$ on $[0, 6]$.

Solution The extreme values occur at critical points or endpoints by Theorem 3, so we can break up the problem neatly into two steps.

Step 1. **Find the critical points.**
The function f is differentiable, so the critical points are solutions to $f'(x) = 0$.

$$f'(x) = 6x^2 - 30x + 24 = 6(x - 1)(x - 4)$$

The critical points satisfy $6(x - 1)(x - 4) = 0$, and therefore are $x = 1$ and 4.

Step 2. **Compare values of $f(x)$ at the critical points and endpoints.**

x-value	Value of $f(x)$	
1 (critical point)	$f(1) = 18$	
4 (critical point)	$f(4) = -9$	min
0 (endpoint)	$f(0) = 7$	
6 (endpoint)	$f(6) = 43$	max

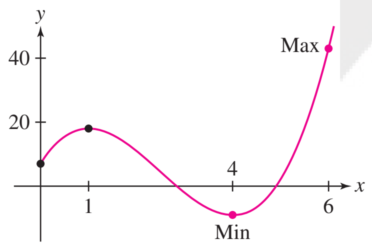


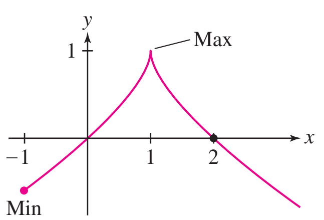
FIGURE 10 Extreme values of $f(x) = 2x^3 - 15x^2 + 24x + 7$ on $[0, 6]$.

The maximum value of $f(x)$ on $[0, 6]$ is the greatest of the values in this table, namely $f(6) = 43$. Similarly, the minimum is $f(4) = -9$. See Figure 10. ■

EXAMPLE 4 Function with a Cusp Find the extrema of $f(x) = 1 - (x - 1)^{2/3}$ on $[-1, 2]$.

Solution First, find the critical points:

$$f'(x) = -\frac{2}{3}(x - 1)^{-1/3} = -\frac{2}{3(x - 1)^{1/3}}$$



DF **FIGURE 11** Extreme values of $f(x) = 1 - (x - 1)^{2/3}$ on $[-1, 2]$.

The equation $f'(x) = 0$ has no solutions because $f'(x)$ is never zero. However, $f'(x)$ does not exist at $x = 1$, so there is a critical point there (Figure 11).

Next, compare values of $f(x)$ at the critical points and endpoints:

x-value	Value of $f(x)$	
1 (critical point)	$f(1) = 1$	max
-1 (endpoint)	$f(-1) \approx -0.59$	min
2 (endpoint)	$f(2) = 0$	

So on $[-1, 2]$, the maximum of f is $f(1) = 1$ and the minimum is $f(-1) \approx -0.59$. ■

EXAMPLE 5 Logarithmic Example Find the extreme values of the function $f(x) = x^2 - 8\ln x$ on $[1, 4]$.

Solution First, we solve for the critical points. We have $f'(x) = 2x - 8/x$, so we solve

$$2x - \frac{8}{x} = 0 \quad \Rightarrow \quad 2x = \frac{8}{x} \quad \Rightarrow \quad x = \pm 2$$

The only critical point in the interval $[1, 4]$ is $x = 2$. Next, compare the values of $f(x)$ at the critical points and endpoints (Figure 12):

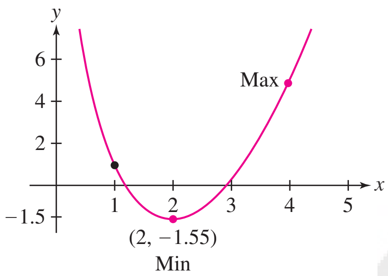


FIGURE 12 Extreme values of $f(x) = x^2 - 8\ln x$ on $[1, 4]$.

x-value	Value of $f(x)$	
2 (critical point)	$f(2) \approx -1.55$	min
1 (endpoint)	$f(1) = 1$	
4 (endpoint)	$f(4) \approx 4.9$	max

We see that the minimum on $[1, 4]$ is $f(2) \approx -1.55$ and the maximum is $f(4) \approx 4.9$. ■

EXAMPLE 6 An Open-Interval Example The function $S(\theta) = 240 + 24\left(\frac{\sqrt{3} - \cos\theta}{\sin\theta}\right)$ arises in a model—that we describe after this example—of the geometry of a honeycomb cell. Figure 13 shows the graph of S for $0 < \theta < \pi$. As θ approaches 0 and π from inside the interval, $S(\theta) \to \infty$. Therefore, there is no absolute maximum of S on $(0, \pi)$, but the graph suggests that there is an absolute minimum. Find it.

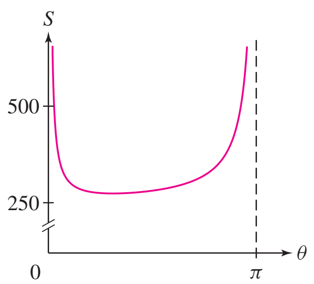


FIGURE 13 Graph of $S(\theta)$.

Solution Computing $S'(\theta)$, we have

$$S'(\theta) = 24\left(\frac{(\sin\theta)(\sin\theta) - (\sqrt{3} - \cos\theta)(\cos\theta)}{\sin^2\theta}\right) = 24\left(\frac{1 - \sqrt{3}\cos\theta}{\sin^2\theta}\right)$$

The derivative is defined for all θ in the interval and is zero where $1 - \sqrt{3}\cos\theta = 0$. Therefore the absolute minimum occurs at

$$\theta_m = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 0.96 \text{ radians} \approx 54.7^\circ$$

Computing $S(0.96)$, we find that the absolute minimum of S over $(0, \pi)$ is approximately 273.94. ■

Using $\theta_m = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$, the minimum value can be shown to be exactly $24(10 + \sqrt{2})$.

Honeycomb Geometry The honeycomb of bees has long been of scientific and mathematical interest (Figure 14). Some believe that, for a fixed cell volume, the specific shape of the cell minimizes the cell's surface area and thus the amount of wax needed to construct it. Each cell has an open hexagonal top, six quadrilateral sides, and three rhombi on the bottom. Imagine that (as in Figure 15) the sides on the hexagonal top are 4 mm, three of the vertical sides are 10 mm, and the remaining dimensions can vary. Let θ be

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FIGURE 14 What is the ideal honeycomb cell shape?

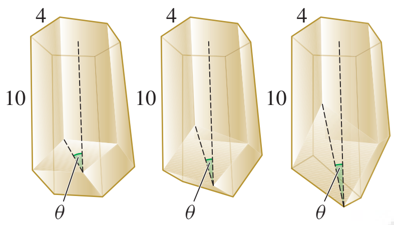


FIGURE 15 The honeycomb cell geometry.

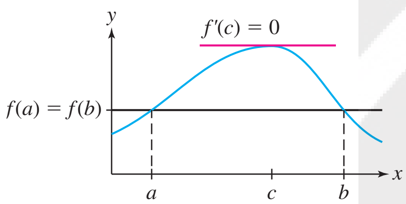


FIGURE 16 Rolle's Theorem: If $f(a) = f(b)$, then $f'(c) = 0$ for some c between a and b.

the angle between a vertical axis through the center of the cell and the bottom rhombi faces. Via geometry, two important facts about the shapes in the figure can be shown:

- The volume of the cell is independent of the angle θ.
- The surface area S of the cell (the total area of the six quadrilaterals and three rhombii) depends on θ according to

$$S(\theta) = 240 + 24\left(\frac{\sqrt{3} - \cos\theta}{\sin\theta}\right)$$

The previous example indicates that for the cells we are considering, the minimum surface area occurs at $\theta \approx 54.7^\circ$. How does this minimum compare with the actual cells that the bees construct?

In the early eighteenth century, astronomer Giacomo Maraldi made extensive measurements of bees' honeycomb and observed typical angle measurements consistent with the optimal angle we found. Subsequently, mathematicians Samuel Koenig and Colin Maclaurin performed a calculus-based analysis of the honeycomb geometry (as we have done here) supporting the idea that the bees are economical in their honeycomb construction. The question of why bees construct the honeycomb as they do is still unsettled, but calculus provides an interesting glimpse at the possibilities.

Rolle's Theorem

As an application of our optimization methods, we prove Rolle's Theorem: If f is differentiable and takes on the same value at two different points a and b, then somewhere between these two points the derivative is zero. Graphically, if the secant line between $x = a$ and $x = b$ is horizontal, then at least one tangent line between a and b is also horizontal (Figure 16).

> **THEOREM 4 Rolle's Theorem** Assume that f is continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then there exists a number c between a and b such that $f'(c) = 0$.

Proof Since f is continuous and $[a, b]$ is closed, f has a min and a max in $[a, b]$. Where do they occur? If either the min or the max occurs at a point c in the *open* interval (a, b), then $f(c)$ is a local extreme value and $f'(c) = 0$ by Fermat's Theorem (Theorem 2). Otherwise, both the min and the max occur at the endpoints. However, $f(a) = f(b)$, so in this case, the min and max coincide and f is a constant function with zero derivative. Then, $f'(c) = 0$ for all c in (a, b). ■

EXAMPLE 7 Illustrating Rolle's Theorem Verify Rolle's Theorem for

$$f(x) = x^4 - x^2 \qquad \text{on} \qquad [-2, 2]$$

Solution The hypotheses of Rolle's Theorem are satisfied because f is differentiable (and therefore continuous) everywhere, and $f(2) = f(-2)$:

$$f(2) = 2^4 - 2^2 = 12, \qquad f(-2) = (-2)^4 - (-2)^2 = 12$$

We must verify that $f'(c) = 0$ has a solution in $(-2, 2)$. Since

$$f'(x) = 4x^3 - 2x = 2x(2x^2 - 1)$$

we need to solve $2x(2x^2 - 1) = 0$. The solutions are $c = 0$ and $c = \pm 1/\sqrt{2} \approx \pm 0.707$. They all lie in $(-2, 2)$, so Rolle's Theorem is satisfied with three values of c. ■

EXAMPLE 8 Using Rolle's Theorem Show that $f(x) = x^3 + 9x - 4$ has precisely one real root.

Solution First, we note that $f(0) = -4$ is negative and $f(1) = 6$ is positive. By the Intermediate Value Theorem (Section 2.8), f has *at least* one root a in $[0, 1]$. If f had a second root b, then we would have $f(a) = f(b) = 0$. Rolle's Theorem would then imply that $f'(c) = 0$ for some $c \in (a, b)$. This is not possible because $f'(x) = 3x^2 + 9 \geq 9 > 0$, so $f'(c) = 0$ has no solutions. We conclude that a is *the only* real root of f (Figure 17). ■

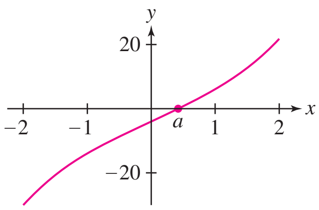
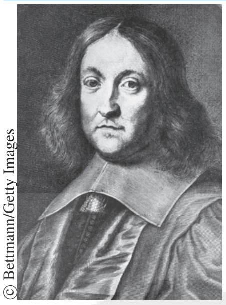


FIGURE 17 Graph of $f(x) = x^3 + 9x - 4$. This function has one real root.

HISTORICAL PERSPECTIVE



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Pierre de Fermat (1601–1665)

© Historical Picture Archive/Getty Images

René Descartes (1596–1650)

Sometime in the 1630s, in the decade before Isaac Newton was born, the French mathematician Pierre de Fermat invented a general method for finding extreme values. Fermat said, in essence, that if you want to find extrema, you must set the derivative equal to zero and solve for the critical points, just as we have done in this section. He also described a general method for finding tangent lines that is not essentially different from our method of derivatives. For this reason, Fermat is often regarded as an inventor of calculus, together with Newton and Leibniz.

At around the same time, René Descartes (1596–1650) developed a different but less effective approach to finding tangent lines. Descartes, after whom Cartesian coordinates are named, was a profound thinker—the leading philosopher and scientist of his time in Europe. He is regarded today as the father of modern philosophy and the founder (along with Fermat) of analytic geometry. A dispute developed when Descartes learned through an intermediary that Fermat had criticized his work on optics. Sensitive and stubborn, Descartes retaliated by attacking Fermat's method of finding tangents, and only after some third-party refereeing did he admit that Fermat was correct. He wrote:

. . . Seeing the last method that you use for finding tangents to curved lines, I can reply to it in no other way than to say that it is very good and that, if you had explained it in this manner at the outset, I would have not contradicted it at all.

However, in subsequent private correspondence, Descartes was less generous, referring at one point to some of Fermat's work as "*le galimatias le plus ridicule*"—meaning the most ridiculous gibberish. Today Fermat is recognized as one of the greatest mathematicians of his age who made far-reaching contributions in several areas of mathematics.

4.2 SUMMARY

- The *extreme values* of f on an interval I are the minimum and maximum values of f for $x \in I$ (also called *absolute extrema* on I).
- Basic Theorem: If f is continuous on a closed interval $[a, b]$, then f has both a min and a max on $[a, b]$.
- $f(c)$ is a *local minimum* if $f(x) \geq f(c)$ for all x in some open interval around c. Local maxima are defined similarly.
- $x = c$ is a *critical point* of f if either $f'(c) = 0$ or $f'(c)$ does not exist.
- Fermat's Theorem on Local Extrema: If $f(c)$ is a local min or max, then c is a critical point.
- To find the extreme values of a continuous function f on a closed interval $[a, b]$:

  Step 1. Find the critical points of f in $[a, b]$.

  Step 2. Calculate $f(x)$ at the critical points in $[a, b]$ and at the endpoints. The min and max on $[a, b]$ are the least and greatest among the values computed in Step 2.

- Rolle's Theorem: If f is continuous on $[a, b]$ and differentiable on (a, b), and if $f(a) = f(b)$, then there exists c between a and b such that $f'(c) = 0$.

4.2 EXERCISES

Preliminary Questions

1. What is the definition of a critical point?

In Questions 2 and 3, which is the correct conclusion, (a) or (b)?

2. If f is not continuous on $[0, 1]$, then

(a) f has no extreme values on $[0, 1]$.

(b) f might not have any extreme values on $[0, 1]$.

3. If f is continuous but has no critical points in $[0, 1]$, then

(a) f has no min or max on $[0, 1]$.

(b) Either $f(0)$ or $f(1)$ is the minimum value on $[0, 1]$.

4. For each statement, indicate whether it is true or false. If false, correct the statement or explain why it is false.

(a) If $f'(c) = 0$, then $f(c)$ is either a local minimum or a local maximum.

(b) If $f(c)$ is the absolute maximum of f on an interval I, then $f'(c) = 0$.

(c) If f is differentiable and $f(c)$ is a local minimum of f, then $f'(c) = 0$.

(d) If there is one local minimum of f on an interval I, then it is the absolute minimum on I.

Exercises

1. The following refer to Figure 18.

(a) What are the critical points of f on $[0, 8]$?

(b) What are the maximum and minimum values of f on $[0, 8]$?

(c) What are the local maximum and minimum values of f, and where do they occur?

(d) Find a closed interval on which both the minimum and maximum values of f occur at critical points.

(e) Find an open interval on which f has neither a minimum nor a maximum value.

(f) Find an open interval on which f has a maximum value but no minimum value.

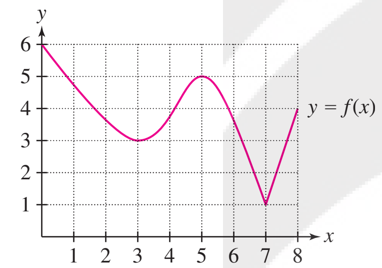


FIGURE 18

2. State whether $f(x) = x^{-1}$ (Figure 19) has a minimum or maximum value on the following intervals:

(a) $(0, 2)$ **(b)** $(1, 2)$ **(c)** $[1, 2]$

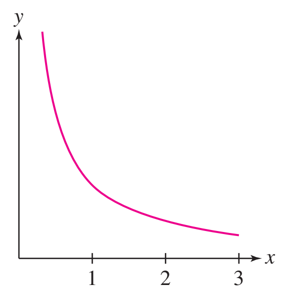


FIGURE 19 Graph of $f(x) = x^{-1}$.

In Exercises 3–20, find all critical points of the function.

3. $f(x) = x^2 - 2x + 4$

4. $f(x) = 7x - 2$

5. $f(x) = x^3 - \frac{9}{2}x^2 - 54x + 2$

6. $f(t) = 8t^3 - t^2$

7. $f(x) = x^{-1} - x^{-2}$

8. $g(z) = \dfrac{1}{z-1} - \dfrac{1}{z}$

9. $f(x) = \dfrac{x}{x^2 + 1}$

10. $f(x) = \dfrac{x^2}{x^2 - 4x + 8}$

11. $f(t) = t - 4\sqrt{t+1}$

12. $f(t) = 4t - \sqrt{t^2 + 1}$

13. $f(x) = xe^{2x}$

14. $f(x) = x + |2x + 1|$

15. $g(\theta) = \sin^2\theta$

16. $R(\theta) = \cos\theta + \sin^2\theta$

17. $f(x) = x \ln x$

18. $f(x) = x^2\sqrt{1 - x^2}$

19. $f(x) = \sin^{-1} x - 2x$

20. $f(x) = \sec^{-1} x - \ln x$

21. Let $f(x) = 2x^2 - 8x + 7$.

(a) Find the critical point c of f and compute $f(c)$.

(b) Find the extreme values of f on $[0, 5]$.

(c) Find the extreme values of f on $[-4, 1]$.

22. Find the extreme values of $f(x) = 2x^3 - 9x^2 + 12x$ on $[0, 3]$ and $[0, 2]$.

23. Find the critical points of $f(x) = \sin x + \cos x$ and determine the extreme values on $\left[0, \frac{\pi}{2}\right]$.

24. Compute the critical points of $h(t) = (t^2 - 1)^{1/3}$. Check that your answer is consistent with Figure 20. Then find the extreme values of h on $[0, 1]$ and on $[0, 2]$.

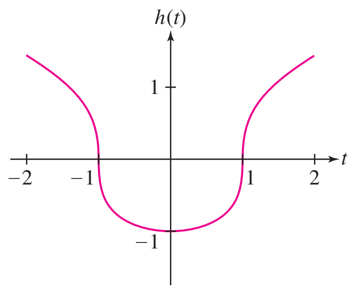


FIGURE 20 Graph of $h(t) = (t^2 - 1)^{1/3}$.

25. GU Plot $f(x) = 4\sqrt{x} - 2x + 3$ on $[0, 3]$ and indicate where it appears that the minimum and maximum occur. Then determine the minimum and maximum using calculus.

26. CAS Approximate the critical points of $g(x) = 5e^x - \tan x$ in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

In Exercises 27–60, find the minimum and maximum values of the function on the given interval by comparing values at the critical points and endpoints.

27. $y = 2x^2 + 4x + 5, \quad [-2, 2]$

28. $y = 2x^2 + 4x + 5, \quad [0, 2]$

29. $y = 6t - t^2, \quad [0, 5]$

30. $y = 6t - t^2, \quad [4, 6]$

31. $y = x^3 - 6x^2 + 8, \quad [1, 6]$

32. $y = x^3 - 6x^2 + 8, \quad [-1, 6]$

33. $y = x^3 - 6x^2 + 8, \quad [1, 3]$

34. $y = x^3 - 6x^2 + 8, \quad [-1, 3]$

35. $y = 2t^3 + 3t^2, \quad [1, 2]$

36. $y = x^3 - 12x^2 + 21x, \quad [0, 2]$

37. $y = z^5 - 80z, \quad [-3, 3]$

38. $y = 2x^5 + 5x^2, \quad [-2, 2]$

39. $y = \dfrac{x^2 + 1}{x - 4}, \quad [5, 6]$

40. $y = \dfrac{1 - x}{x^2 + 3x}, \quad [1, 4]$

41. $y = x - \dfrac{4x}{x + 1}, \quad [0, 3]$

42. $y = 2\sqrt{x^2 + 1} - x, \quad [0, 2]$

43. $y = (2 + x)\sqrt{2 + (2 - x)^2}, \quad [0, 2]$

44. $y = \sqrt{1 + x^2} - 2x, \quad [0, 1]$

45. $y = \sqrt{x + x^2} - 2\sqrt{x}, \quad [0, 4]$

46. $y = (t - t^2)^{1/3}, \quad [-1, 2]$

47. $y = \sin x \cos x, \quad \left[0, \frac{\pi}{2}\right]$

48. $y = x + \sin x, \quad [0, 2\pi]$

49. $y = \sqrt{2}\,\theta - \sec\theta, \quad \left[0, \frac{\pi}{3}\right]$

50. $x^4 - 2x^2 + 1, \quad [-3, 3]$

51. $y = x^3 + x^2 - x, \quad [-2, 2]$

52. $y = \cos\theta + \sin\theta, \quad [0, 2\pi]$

53. $y = \theta - 2\sin\theta, \quad [0, 2\pi]$

54. $y = 4\sin^3\theta - 3\cos^2\theta, \quad [0, 2\pi]$

55. $y = \tan x - 2x, \quad [0, 1]$

56. $y = xe^{-x}, \quad [0, 2]$

57. $y = \dfrac{\ln x}{x}, \quad [1, 3]$

58. $y = 5\tan^{-1} x - x, \quad [1, 5]$

59. $y = 3e^x - e^{2x}, \quad \left[-\frac{1}{2}, 1\right]$

60. $y = x^3 - 24\ln x, \quad \left[\frac{1}{2}, 3\right]$

61. GU Plot $f(x) = \frac{2+x^2}{x}$ on $(0, 5)$ and use the graph to explain why there is a minimum value, but no maximum value, of f on $(0, 5)$. Use calculus to find the minimum value.

62. GU Plot $f(x) = \frac{4x-1-x^2}{x}$ on $(0, 3)$ and use the graph to explain why there is a maximum value, but no minimum value, of f on $(0, 3)$. Use calculus to find the maximum value.

63. Let $f(\theta) = 2\sin 2\theta + \sin 4\theta$.

(a) Show that θ is a critical point if $\cos 4\theta = -\cos 2\theta$.

(b) Show, using a unit circle, that $\cos\theta_1 = -\cos\theta_2$ if and only if $\theta_1 = \pi \pm \theta_2 + 2\pi k$ for an integer k.

(c) Show that $\cos 4\theta = -\cos 2\theta$ if and only if $\theta = \frac{\pi}{2} + \pi k$ or $\theta = \frac{\pi}{6} + \left(\frac{\pi}{3}\right)k$.

(d) Find the six critical points of f on $[0, 2\pi]$ and find the extreme values of f on this interval.

(e) GU Check your results against a graph of f.

64. GU Find the critical points of $f(x) = 2\cos 3x + 3\cos 2x$ in $[0, 2\pi]$. Check your answer against a graph of f.

In Exercises 65–68, find the critical points and the extreme values on $[0, 4]$. *In Exercises 67 and 68, refer to Figure 21.*

65. $y = |x - 2|$

66. $y = |3x - 9|$

67. $y = |x^2 + 4x - 12|$

68. $y = |\cos x|$

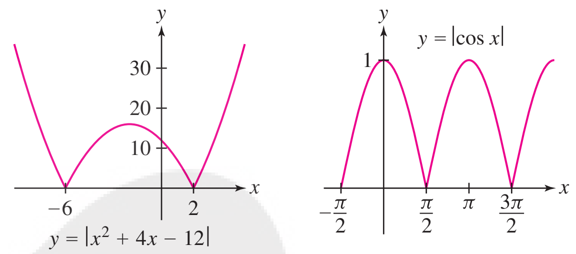


FIGURE 21

In Exercises 69–72, verify Rolle's Theorem for the given interval by checking $f(a) = f(b)$ *and then finding a value* c *in* (a, b) *such that* $f'(c) = 0$.

69. $f(x) = x + x^{-1}, \quad \left[\frac{1}{2}, 2\right]$

70. $f(x) = \sin x, \quad \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$

71. $f(x) = \dfrac{x^2}{8x - 15}, \quad [3, 5]$

72. $f(x) = \sin^2 x - \cos^2 x, \quad \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$

73. Prove that $f(x) = x^5 + 2x^3 + 4x - 12$ has precisely one real root.

74. Prove that $f(x) = x^3 + 3x^2 + 6x$ has precisely one real root.

75. Prove that $f(x) = x^4 + 5x^3 + 4x$ has no root c satisfying $c > 0$. *Hint:* Note that $x = 0$ is a root and apply Rolle's Theorem.

76. Prove that $x = 4$ is the greatest root of $f(x) = x^4 - 8x^2 - 128$.

77. The position of a mass oscillating at the end of a spring is $s(t) = A\sin\omega t$, where A is the amplitude and ω is the angular frequency. Show that the speed $|v(t)|$ is at a maximum when the acceleration $a(t)$ is zero and that $|a(t)|$ is at a maximum when $v(t)$ is zero.

78. The concentration $C(t)$ (in milligrams per cubic centimeter) of a drug in a patient's bloodstream after t hours is

$$C(t) = \frac{0.016t}{t^2 + 4t + 4}$$

Find the maximum concentration in the time interval $[0, 8]$ and the time at which it occurs.

79. CAS **Antibiotic Levels** A study shows that the concentration $C(t)$ (in micrograms per milliliter) of antibiotic in a patient's blood serum after t hours is $C(t) = 120(e^{-0.2t} - e^{-bt})$, where $b \geq 1$ is a constant that depends on the particular combination of antibiotic agents used. Solve numerically for the value of b (to two decimal places) for which maximum concentration occurs at $t = 1$ h. You may assume that the maximum occurs at a critical point, as suggested by Figure 22.

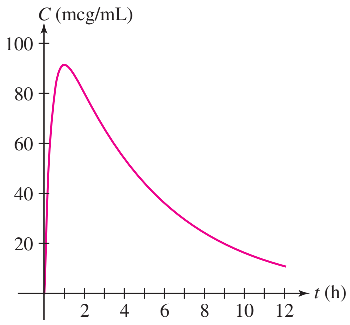


FIGURE 22 Graph of $C(t) = 120(e^{-0.2t} - e^{-bt})$ with b chosen so that the maximum occurs at $t = 1$ h.

80. CAS In the notation of Exercise 79, find the value of b (to two decimal places) for which the maximum value of C is equal to 100 mcg/ml.

81. In 1919, physicist Alfred Betz argued that the maximum efficiency of a wind turbine is around 59%. If wind enters a turbine with speed v_1 and exits with speed v_2, then the power extracted is the difference in kinetic energy per unit time:

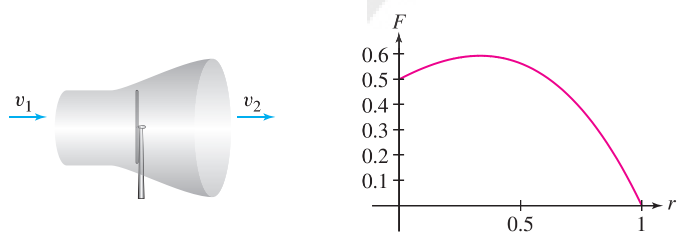
$$P = \frac{1}{2}mv_1^2 - \frac{1}{2}mv_2^2 \quad \text{watts}$$

where m is the mass of wind flowing through the rotor per unit time (Figure 23). Betz assumed that $m = \rho A(v_1 + v_2)/2$, where ρ is the density of air and A is the area swept out by the rotor. Wind flowing undisturbed through the same area A would have mass per unit time $\rho A v_1$ and power $P_0 = \frac{1}{2}\rho A v_1^3$. The fraction of power extracted by the turbine is $F = P/P_0$.

(a) Show that F depends only on the ratio $r = v_2/v_1$ and is equal to $F(r) = \frac{1}{2}(1 - r^2)(1 + r)$, where $0 \le r \le 1$.

(b) Show that the maximum value of F, called the **Betz Limit**, is $16/27 \approx 0.59$.

(c) Explain why Betz's formula for F is not meaningful for r close to zero. *Hint:* How much wind would pass through the turbine if v_2 were zero? Is this realistic?



(A) Wind flowing through a turbine.

(B) F is the fraction of energy extracted by the turbine as a function of $r = v_2/v_1$.

FIGURE 23

82. GU The **Bohr radius** a_0 of the hydrogen atom is the value of r that minimizes the energy

$$E(r) = \frac{\hbar^2}{2mr^2} - \frac{e^2}{4\pi\epsilon_0 r}$$

where $\hbar$, m, e, and ϵ_0 are physical constants. Show that $a_0 = 4\pi\epsilon_0\hbar^2/(me^2)$. Assume that the minimum occurs at a critical point, as suggested by Figure 24.

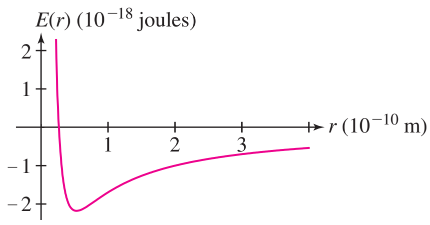
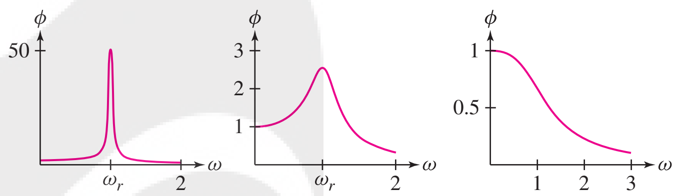


FIGURE 24

83. The response of a circuit or other oscillatory system to an input of frequency ω ("omega") is described by the function

$$\phi(\omega) = \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4D^2\omega^2}}$$

Both ω_0 (the natural frequency of the system) and D (the damping factor) are positive constants. The graph of ϕ is called a **resonance curve**, and the positive frequency $\omega_r > 0$, where ϕ takes its maximum value, if it exists, is called the **resonant frequency**. Show that $\omega_r = \sqrt{\omega_0^2 - 2D^2}$ if $0 < D < \omega_0/\sqrt{2}$ and that no resonant frequency exists otherwise (Figure 25).



(A) $D = 0.01$ (B) $D = 0.2$ (C) $D = 0.75$ (no resonance)

FIGURE 25 Resonance curves with $\omega_0 = 1$.

84. Find the maximum and minimum of $y = (1 - x)e^{ax}$ on $[0, 1]$, where $0 < a$.

85. Find the maximum of $y = x^a - x^b$ on $[0, 1]$, where $0 < a < b$. In particular, find the maximum of $y = x^5 - x^{10}$ on $[0, 1]$.

In Exercises 86–88, plot the function using a graphing utility and find its critical points and extreme values on $[-5, 5]$.

86. GU $y = \dfrac{1}{1 + |x - 1|}$

87. GU $y = \dfrac{1}{1 + |x - 1|} + \dfrac{1}{1 + |x - 4|}$

88. GU $y = \dfrac{x}{|x^2 - 1| + |x^2 - 4|}$

89. **(a)** Use implicit differentiation to find the critical points on the curve $27x^2 = (x^2 + y^2)^3$.

(b) GU Plot the curve and the horizontal tangent lines on the same set of axes.

90. Sketch the graph of a continuous function on $(0, 4)$ with a minimum value but no maximum value.

91. Sketch the graph of a continuous function on $(0, 4)$ having a local minimum but no absolute minimum.

92. Sketch the graph of a function on $[0, 4]$ having

(a) Two local maxima and one local minimum

(b) An absolute minimum that occurs at an endpoint, and an absolute maximum that occurs at a critical point

93. Sketch the graph of a function f on $[0, 4]$ with a discontinuity such that f has an absolute minimum but no absolute maximum.

94. A rainbow is produced by light rays that enter a raindrop (assumed spherical) and exit after being reflected internally as in Figure 26. The angle between the incoming and reflected rays is $\theta = 4r - 2i$, where the angle of incidence i and the angle of refraction r are related by Snell's Law $\sin i = n \sin r$ with $n \approx 1.33$ (the index of refraction for air and water).

(a) Use Snell's Law to show that $\dfrac{dr}{di} = \dfrac{\cos i}{n \cos r}$.

(b) Show that the maximum value $\theta_{\max}$ of θ occurs when i satisfies $\cos i = \sqrt{\dfrac{n^2-1}{3}}$. *Hint:* Show that $\dfrac{d\theta}{di} = 0$ if $\cos i = \dfrac{n}{2}\cos r$. Then use Snell's Law to eliminate r.

(c) Show that $\theta_{\max} \approx 42.53°$.

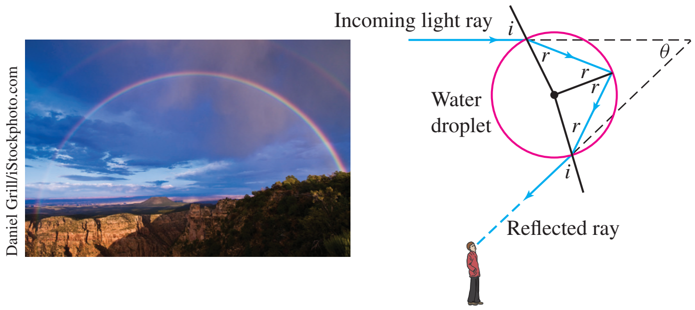


FIGURE 26

Further Insights and Challenges

95. Show that the extreme values of $f(x) = a \sin x + b \cos x$ are $\pm\sqrt{a^2 + b^2}$.

96. Show, by considering its minimum, that $f(x) = x^2 - 2x + 3$ takes on only positive values. More generally, find the conditions on r and s under which the quadratic function $f(x) = x^2 + rx + s$ takes on only positive values. Give examples of r and s for which f takes on both positive and negative values.

97. Show that if the quadratic polynomial $f(x) = x^2 + rx + s$ takes on both positive and negative values, then its minimum value occurs at the midpoint between the two roots.

98. Generalize Exercise 97: Show that if the horizontal line $y = c$ intersects the graph of $f(x) = x^2 + rx + s$ at two points $(x_1, f(x_1))$ and $(x_2, f(x_2))$, then f takes its minimum value at the midpoint $M = \dfrac{x_1 + x_2}{2}$ (Figure 27).

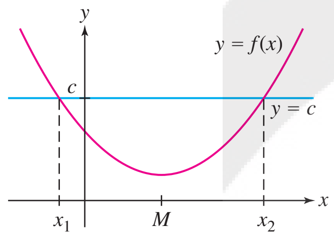


FIGURE 27

99. A cubic polynomial may have a local min and max, or it may have neither (Figure 28). Find conditions on the coefficients a and b of

$$f(x) = \frac{1}{3}x^3 + \frac{1}{2}ax^2 + bx + c$$

that ensure f has neither a local min nor a local max. *Hint:* Apply Exercise 96 to $f'(x)$.

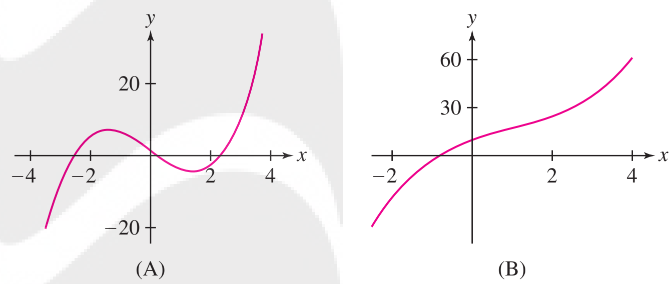


FIGURE 28 Cubic polynomials.

100. Find the min and max of

$$f(x) = x^p(1-x)^q \quad \text{on } [0, 1]$$

where $p, q > 0$.

101. Prove that if f is continuous and $f(a)$ and $f(b)$ are local minima where $a < b$, then there exists a value c between a and b such that $f(c)$ is a local maximum. (*Hint:* Apply Theorem 1 to the interval $[a, b]$.) Show that continuity is a necessary hypothesis by sketching the graph of a function (necessarily discontinuous) with two local minima but no local maximum.

4.3 The Mean Value Theorem and Monotonicity

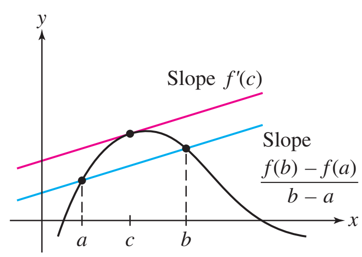


FIGURE 1 By the MVT, there exists at least one tangent line parallel to the secant line.

We have taken for granted that if $f'(x)$ is positive, the function f is increasing, and if $f'(x)$ is negative, f is decreasing. In this section, we prove this rigorously using an important result called the Mean Value Theorem (MVT). Then we develop a method for "testing" critical points—that is, for determining whether they correspond to local maxima, local minima, or neither.

The MVT says that a secant line between two points $(a, f(a))$ and $(b, f(b))$ on a graph is parallel to at least one tangent line in the interval (a, b) (Figure 1). Since the secant line between $(a, f(a))$ and $(b, f(b)$ has slope $\dfrac{f(b) - f(a)}{b - a}$ and since two lines are parallel if they have the same slope, the MVT is claiming that there exists a point c between a and b such that

$$\underbrace{f'(c)}_{\text{Slope of tangent line}} = \underbrace{\frac{f(b)-f(a)}{b-a}}_{\text{Slope of secant line}}$$

THEOREM 1 The Mean Value Theorem Assume that f is continuous on the closed interval $[a,b]$ and differentiable on (a,b). Then there exists at least one value c in (a,b) such that

$$\boxed{f'(c) = \frac{f(b)-f(a)}{b-a}}$$

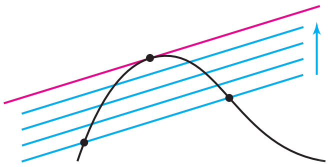


FIGURE 2 Move the secant line in a parallel fashion until it becomes tangent to the curve.

Rolle's Theorem (Section 4.2) is the special case of the MVT in which $f(a) = f(b)$. In this case, the conclusion is that $f'(c) = 0$.

GRAPHICAL INSIGHT Imagine what happens when a secant line is moved parallel to itself. Eventually, it becomes a tangent line, as shown in Figure 2. This is the idea behind the MVT. We present a formal proof at the end of this section.

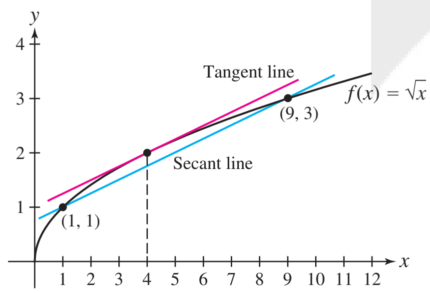
CONCEPTUAL INSIGHT The conclusion of the MVT can be rewritten as

$$\boxed{f(b) - f(a) = f'(c)(b-a)}$$

We can think of this as a variation on the Linear Approximation, which says

$$f(b) - f(a) \approx f'(a)(b-a)$$

The MVT turns this approximation into an equality by replacing $f'(a)$ with $f'(c)$ for a suitable choice of c in (a,b).



DF FIGURE 3 The tangent line at $c = 4$ is parallel to the secant line.

EXAMPLE 1 Verify the MVT with $f(x) = \sqrt{x}$, $a = 1$, and $b = 9$.

Solution First, compute the slope of the secant line (Figure 3):

$$\frac{f(b)-f(a)}{b-a} = \frac{\sqrt{9}-\sqrt{1}}{9-1} = \frac{3-1}{9-1} = \frac{1}{4}$$

We must find c such that $f'(c) = 1/4$. The derivative is $f'(x) = \frac{1}{2}x^{-1/2}$, and

$$f'(c) = \frac{1}{2\sqrt{c}} = \frac{1}{4} \quad \Rightarrow \quad 2\sqrt{c} = 4 \quad \Rightarrow \quad c = 4$$

The value $c = 4$ lies in $(1,9)$ and satisfies $f'(4) = \frac{1}{4}$. This verifies the MVT. ■

As a first application, we prove that a function with zero derivative is constant.

COROLLARY If f is differentiable and $f'(x) = 0$ for all $x \in (a,b)$, then f is constant on (a,b). In other words, $f(x) = C$ for some constant C.

Proof If a_1 and b_1 are any two distinct points in (a,b), then, by the MVT, there exists c between a_1 and b_1 such that

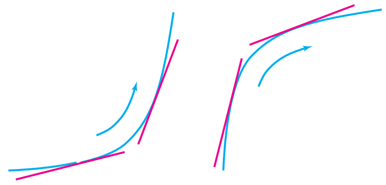
$$f(b_1) - f(a_1) = f'(c)(b_1 - a_1) = 0 \qquad (\text{since } f'(c) = 0)$$

Thus, $f(b_1) = f(a_1)$. This says that $f(x)$ is constant on (a,b). ■

We say that f is "nondecreasing" if

$$f(x_1) \le f(x_2) \quad \text{for} \quad x_1 \le x_2$$

"Nonincreasing" is defined similarly. In Theorem 2, if we assume that $f'(x) \ge 0$ (instead of > 0), then f is nondecreasing on (a, b). If $f'(x) \le 0$, then f is nonincreasing on (a, b).



Increasing function:
Tangent lines have positive slope.

Decreasing function:
Tangent lines have negative slope.

FIGURE 4

Increasing/Decreasing Behavior of Functions

We prove now that the sign of the derivative determines whether a function f is increasing or decreasing. Recall that f is

- **Increasing on** (a, b) if $f(x_1) < f(x_2)$ for all $x_1, x_2 \in (a, b)$ such that $x_1 < x_2$.
- **Decreasing on** (a, b) if $f(x_1) > f(x_2)$ for all $x_1, x_2 \in (a, b)$ such that $x_1 < x_2$.

We say that f is **monotonic** on (a, b) if it is either increasing or decreasing on (a, b).

THEOREM 2 The Sign of the Derivative Let f be a differentiable function on an open interval (a, b).

- If $f'(x) > 0$ for $x \in (a, b)$, then f is increasing on (a, b).
- If $f'(x) < 0$ for $x \in (a, b)$, then f is decreasing on (a, b).

Proof Suppose first that $f'(x) > 0$ for all $x \in (a, b)$. The MVT tells us that for any two points $x_1 < x_2$ in (a, b), there exists c between x_1 and x_2 such that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1) > 0$$

The inequality holds because $f'(c)$ and $(x_2 - x_1)$ are both positive. Thus, $f(x_2) > f(x_1)$, as required. The case $f'(x) < 0$ is similar. ∎

GRAPHICAL INSIGHT Theorem 2 confirms our graphical intuition (Figure 4):

- $f'(x) > 0 \quad \Rightarrow$ tangent lines have positive slope $\Rightarrow$ f increasing
- $f'(x) < 0 \quad \Rightarrow$ tangent lines have negative slope $\Rightarrow$ f decreasing

EXAMPLE 2 Show that $f(x) = \ln x$ is increasing.

Solution The derivative $f'(x) = x^{-1}$ is positive on the domain $\{x : x > 0\}$, so $f(x) = \ln x$ is increasing (Figure 5). ∎

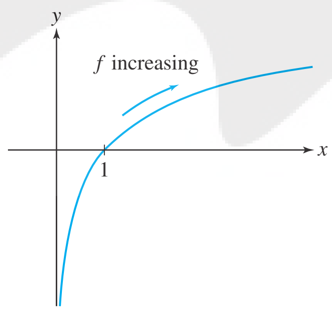


FIGURE 5 Graph of $f(x) = \ln x$.

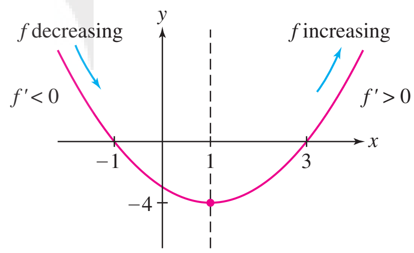


FIGURE 6 Graph of $f(x) = x^2 - 2x - 3$.

EXAMPLE 3 Find the intervals on which $f(x) = x^2 - 2x - 3$ is monotonic.

Solution The derivative $f'(x) = 2x - 2 = 2(x - 1)$ is positive for $x > 1$ and negative for $x < 1$. By Theorem 2, f is decreasing on the interval $(-\infty, 1)$ and increasing on the interval $(1, \infty)$, as confirmed in Figure 6. ∎

Testing Critical Points

There is a useful test for determining whether a critical point yields a min or max (or neither) based on the *sign change* of the derivative $f'(x)$.

To explain the term "sign change," suppose that a function g satisfies $g(c) = 0$. We say that $g(x)$ *changes from positive to negative* at $x = c$ if $g(x) > 0$ to the left of

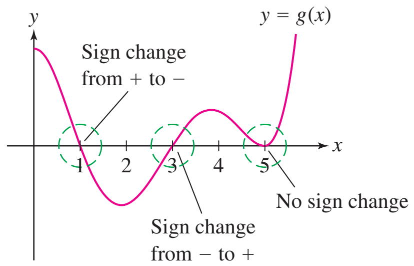
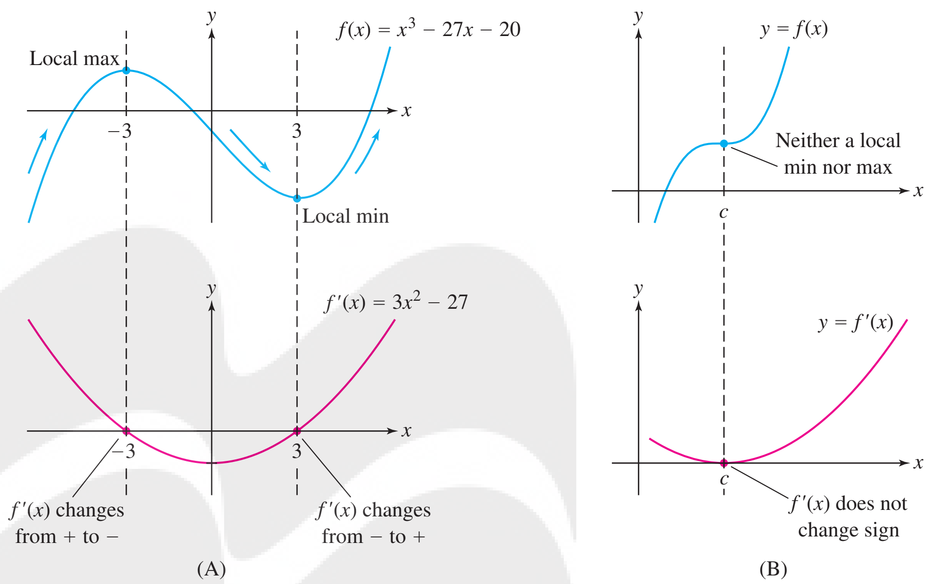


FIGURE 7

c and $g(x) < 0$ to the right of c for x within a small open interval around c (Figure 7). A sign change from negative to positive is defined similarly. Observe in Figure 7 that $g(5) = 0$ but $g(x)$ does not change sign at $x = 5$.

Now suppose that $f'(c) = 0$ and that $f'(x)$ changes sign at $x = c$, say, from $+$ to $-$. Then f is increasing to the left of c and decreasing to the right, so $f(c)$ is a local maximum. Similarly, if $f'(x)$ changes sign from $-$ to $+$, then $f(c)$ is a local minimum. See Figure 8(A). Figure 8(B) illustrates a case where $f'(c) = 0$ but $f'(x)$ does not change sign. In this case, $f'(x) > 0$ for all x near but not equal to c, so f is increasing and has neither a local min nor a local max at c.



DF FIGURE 8

A similar analysis holds when $f'(c)$ does not exist and the possibilities for the sign of f' on either side of c are considered. As a result, we have the following theorem:

THEOREM 3 First Derivative Test for Critical Points Let c be a critical point of f. Then

- $f'(x)$ changes from $+$ to $-$ at c $\Rightarrow$ $f(c)$ is a local maximum.
- $f'(x)$ changes from $-$ to $+$ at c $\Rightarrow$ $f(c)$ is a local minimum.

To carry out the First Derivative Test, we make a useful observation: $f'(x)$ can change sign at a critical point, but it cannot change sign on the interval between two consecutive critical points as long as the function is defined over the whole interval. In such a case, we can determine the sign of $f'(x)$ on an interval between consecutive critical points by evaluating $f'(x)$ at any *test point* x_0 inside the interval. The sign of $f'(x_0)$ is the sign of $f'(x)$ on the entire interval. In a case where a function's domain is made up of separate intervals, this analysis of the sign of f' needs to be carried out individually on each of the intervals.

EXAMPLE 4 Analyze the critical points of $f(x) = x^3 - 27x - 20$.

Solution Our analysis will confirm the picture in Figure 8(A).

Step 1. **Find the critical points.**
We have $f'(x) = 3x^2 - 27 = 3(x^2 - 9)$. The critical points satisfy $f'(c) = 0$ and therefore are $c = \pm 3$.

Step 2. **Find the sign of $f'(x)$ on the intervals between the critical points.**
The critical points $c = \pm 3$ divide the real line into three intervals:

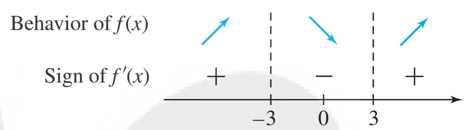
$$(-\infty, -3), \qquad (-3, 3), \qquad (3, \infty)$$

To determine the sign of $f'(x)$ on these intervals, we choose a test point inside each interval and evaluate. For example, in $(-\infty, -3)$ we choose $x = -4$. Because $f'(-4) = 21 > 0$, $f'(x)$ is positive on the entire interval $(-3, \infty)$. Taking this result, along with the results from test points at 0 and 4, we have

We chose the test points -4, 0, and 4 arbitrarily. To find the sign of $f'(x)$ on $(-\infty, -3)$, we could just as well have computed $f'(-5)$ or any other value of f' in the interval $(-\infty, -3)$.

$$\begin{aligned} f'(-4) &= 21 > 0 &\quad\Rightarrow\quad& f'(x) > 0 \quad \text{for all } x \in (-\infty, -3) \\ f'(0) &= -27 < 0 &\quad\Rightarrow\quad& f'(x) < 0 \quad \text{for all } x \in (-3, 3) \\ f'(4) &= 21 > 0 &\quad\Rightarrow\quad& f'(x) > 0 \quad \text{for all } x \in (3, \infty) \end{aligned}$$

This information is displayed in the following sign diagram:



Step 3. **Use the First Derivative Test.**

- $c = -3$: $f'(x)$ changes from $+$ to $-$ $\Rightarrow$ $f(-3) = 34$ is a local maximum value.
- $c = 3$: $f'(x)$ changes from $-$ to $+$ $\Rightarrow$ $f(3) = -74$ is a local minimum value. ■

EXAMPLE 5 Analyze the critical points and the increase/decrease behavior of $f(x) = \cos^2 x + \sin x$ in $(0, \pi)$.

Solution First, find the critical points:

$$f'(x) = -2\cos x \sin x + \cos x = (\cos x)(1 - 2\sin x)$$

Therefore, the critical points are solutions to $\cos x = 0$ or $\sin x = \frac{1}{2}$. Since we are just examining the interval $(0, \pi)$, the critical points of interest are $\frac{\pi}{6}$, $\frac{\pi}{2}$, and $\frac{5\pi}{6}$. They divide $(0, \pi)$ into four intervals:

$$\left(0, \frac{\pi}{6}\right), \qquad \left(\frac{\pi}{6}, \frac{\pi}{2}\right), \qquad \left(\frac{\pi}{2}, \frac{5\pi}{6}\right), \qquad \left(\frac{5\pi}{6}, \pi\right)$$

We determine the sign of $f'(x)$ by evaluating $f'(x)$ at a test point inside each interval. Since $\frac{\pi}{6} \approx 0.52$, $\frac{\pi}{2} \approx 1.57$, $\frac{5\pi}{6} \approx 2.62$, and $\pi \approx 3.14$, we can use the following test points:

Interval	Test value	Sign of $f'(x)$	Behavior of $f(x)$
$(0, \frac{\pi}{6})$	$f'(0.5) \approx 0.04$	$+$	↗
$(\frac{\pi}{6}, \frac{\pi}{2})$	$f'(1) \approx -0.37$	$-$	↘
$(\frac{\pi}{2}, \frac{5\pi}{6})$	$f'(2) \approx 0.34$	$+$	↗
$(\frac{5\pi}{6}, \pi)$	$f'(3) \approx -0.71$	$-$	↘

Now apply the First Derivative Test:

- Local max at $c = \frac{\pi}{6}$ and $c = \frac{5\pi}{6}$ because $f'(x)$ changes from $+$ to $-$.
- Local min at $c = \frac{\pi}{2}$ because $f'(x)$ changes from $-$ to $+$.

The behavior of $f(x)$ and $f'(x)$ is reflected in the graphs in Figure 9. ■

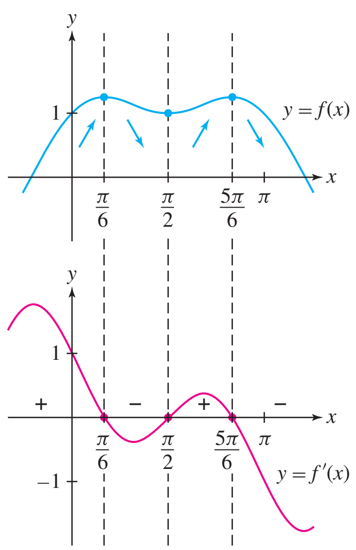


FIGURE 9 Graph of $f(x) = \cos^2 x + \sin x$ and its derivative.

EXAMPLE 6 Analyze the critical points and the increase/decrease behavior of $f(x) = x^2 + \frac{1}{x^2}$.

Solution Note that f is undefined at $x = 0$, so we need to analyze f separately on $(-\infty, 0)$ and $(0, \infty)$. We have

$$f'(x) = 2x - \frac{2}{x^3}$$

The critical points are solutions to $x - \frac{1}{x^3} = 0$; that is, to $x^4 - 1 = 0$. They are $c = \pm 1$. Since we need to consider f separately on $(-\infty, 0)$ and $(0, \infty)$, there are four intervals on which we need to examine the sign of $f'(x)$: $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$, and $(1, \infty)$. We determine the sign of $f'(x)$ by evaluating $f'(x)$ at a test point inside each interval.

Interval	Test value	Sign of $f'(x)$	Behavior of $f(x)$
$(-\infty, -1)$	$f'(-2) = -3.75$	$-$	$\searrow$
$(-1, 0)$	$f'(-0.5) = 15$	$+$	$\nearrow$
$(0, 1)$	$f'(0.5) = -15$	$-$	$\searrow$
$(1, \infty)$	$f'(2) = 3.75$	$+$	$\nearrow$

Applying the First Derivative Test, we see that both critical points are local minima. This is verified in the graph in Figure 10. ■

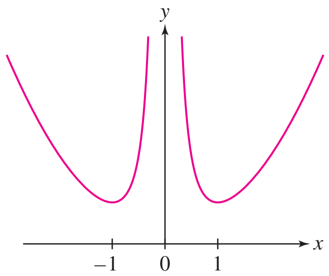


FIGURE 10 Graph of $f(x) = x^2 + \frac{1}{x^2}$.

EXAMPLE 7 **A Critical Point Where $f'(x)$ Is Undefined** Analyze the critical points of $f(x) = (1 - x)^{2/3}$.

Solution The derivative is $f'(x) = -\frac{2}{3}(1 - x)^{-1/3} = \frac{-2}{3(1-x)^{1/3}}$. The only critical point occurs at $c = 1$, when $f'(x)$ is undefined. For $x < 1$, $f'(x)$ is negative. For $x > 1$, $f'(x)$ is positive. So $f'(x)$ changes sign as we pass through $c = 1$, and by the First Derivative Test, $f(c)$ is a local minimum. See Figure 11. ■

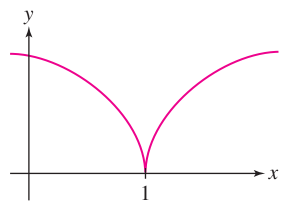


FIGURE 11 Graph of $f(x) = (1 - x)^{2/3}$.

EXAMPLE 8 **Infinitely Many Critical Points, No Local Extrema** Analyze the critical points of $f(x) = x - \sin x$.

Solution We have $f'(x) = 1 - \cos x$, and therefore critical points occur at solutions to $\cos x = 1$; that is, at $n\pi$ for all even integers n. At none of the critical points does the sign of f' change since $f'(x) \geq 0$ for all x. Therefore, none of the critical points are local extrema (Figure 12). ■

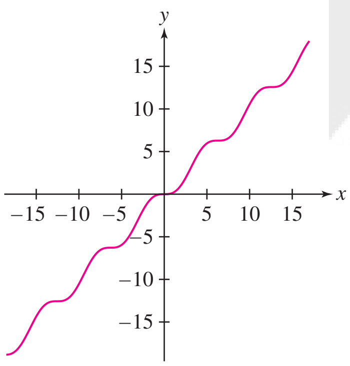


FIGURE 12 Graph of $f(x) = x - \sin x$.

Proof of the MVT Let $m = \dfrac{f(b) - f(a)}{b - a}$ be the slope of the secant line joining $(a, f(a))$ and $(b, f(b))$. The secant line has equation $y = mx + r$ for some r (Figure 13). Now consider the function

$$G(x) = f(x) - (mx + r)$$

As indicated in Figure 13, $G(x)$ is the vertical distance between the graph and the secant line at x (it is negative at points where the graph of f lies below the secant line). This distance is zero at the endpoints, and therefore, $G(a) = G(b) = 0$. By Rolle's Theorem (Section 4.2), there exists a point c in (a, b) such that $G'(c) = 0$. But $G'(x) = f'(x) - m$, so $G'(c) = f'(c) - m = 0$, and $f'(c) = m$ as desired. ■

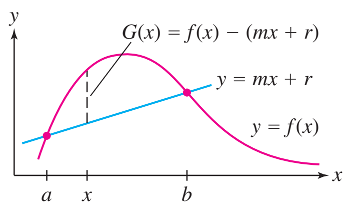


FIGURE 13 $G(x)$ is the vertical distance between the graph and the secant line.

4.3 SUMMARY

- The Mean Value Theorem (MVT): If f is continuous on $[a, b]$ and differentiable on (a, b), then there exists at least one value c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

This conclusion can also be written

$$f(b) - f(a) = f'(c)(b - a)$$

- Important corollary of the MVT: If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on (a, b).
- The *sign* of $f'(x)$ determines whether f is increasing or decreasing:

$$f'(x) > 0 \text{ for } x \in (a, b) \quad \Rightarrow \quad f \text{ is increasing on } (a, b)$$
$$f'(x) < 0 \text{ for } x \in (a, b) \quad \Rightarrow \quad f \text{ is decreasing on } (a, b)$$

- On an interval over which f is defined, the sign of $f'(x)$ can change only at the critical points, so f is *monotonic* (increasing or decreasing) on the intervals between the critical points.
- On an interval over which f is defined, to find the sign of $f'(x)$ on an interval between two critical points, calculate the sign of $f'(x_0)$ at any test point x_0 in that interval.
- First Derivative Test: If f is differentiable and c is a critical point, then

Sign change of $f'(x)$ at c	**Type of critical point**
From + to −	Local maximum
From − to +	Local minimum

4.3 EXERCISES

Preliminary Questions

1. For which value of m is the following statement correct? If $f(2) = 3$ and $f(4) = 9$, and f is differentiable, then f has a tangent line of slope m.

2. Assume f is differentiable. Which of the following statements does *not* follow from the MVT?

(a) If f has a secant line of slope 0, then f has a tangent line of slope 0.

(b) If $f(5) < f(9)$, then $f'(c) > 0$ for some $c \in (5, 9)$.

(c) If f has a tangent line of slope 0, then f has a secant line of slope 0.

(d) If $f'(x) > 0$ for all x, then every secant line has positive slope.

3. Can a function with the real numbers as its domain that takes on only negative values have a positive derivative? If so, sketch an example.

4. For f with derivative as in Figure 14:

(a) Is $f(c)$ a local minimum or maximum?

(b) Is f a decreasing function?

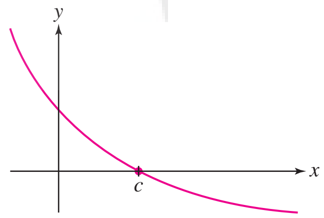


FIGURE 14 Graph of derivative f'.

5. Which of the six standard trigonometric functions have infinitely many local minima and infinitely many local maxima but no absolute maximum and no absolute minimum over their whole domain?

6. Compose the absolute value with a familiar function to define a function f that

- has infinitely many local maxima, all of which occur where $f' = 0$, and
- has infinitely many local minima, all of which occur where f' is undefined.

Exercises

In Exercises 1–8, find a point c satisfying the conclusion of the MVT for the given function and interval.

1. $y = x^{-1}$, $[2, 8]$

2. $y = \sqrt{x}$, $[9, 25]$

3. $y = \cos x - \sin x$, $[0, 2\pi]$

4. $y = \dfrac{x}{x+2}$, $[1, 4]$

5. $y = x^3$, $[-4, 5]$

6. $y = x \ln x$, $[1, 2]$

7. $y = e^{-2x}$, $[0, 3]$

8. $y = e^x - x$, $[-1, 1]$

In Exercises 9–12, find a point c satisfying the conclusion of the MVT for the given function and interval. Then draw the graph of the function, the secant line between the endpoints of the graph and the tangent line at $(c, f(c))$, to see that the secant and tangent lines are, in fact, parallel.

9. $y = x^2$, $[0, 1]$

10. $y = x^{2/3}$, $[0, 8]$

11. $y = e^x$, $[0, 1]$

12. $y = \sqrt{x}$, $[0, 3]$

13. GU Let $f(x) = x^5 + x^2$. The secant line between $(0, 0)$ and $(1, 2)$ has slope 2 (check this), so by the MVT, $f'(c) = 2$ for some $c \in (0, 1)$. Plot f and the secant line on the same axes. Then plot $y = 2x + b$ for different values of b until the line becomes tangent to the graph of f. Zoom in on the point of tangency to estimate the x-coordinate c of the point of tangency.

14. GU Plot the derivative of $f(x) = 3x^5 - 5x^3$. Describe its sign changes and use this to determine the local extreme values of f. Then graph f to confirm your conclusions.

15. Determine the intervals on which $f'(x)$ is positive and negative, assuming that Figure 15 is the graph of f.

16. Determine the intervals on which f is increasing or decreasing, assuming that Figure 15 is the graph of f'.

17. State whether $f(2)$ and $f(4)$ are local minima or local maxima, assuming that Figure 15 is the graph of f'.

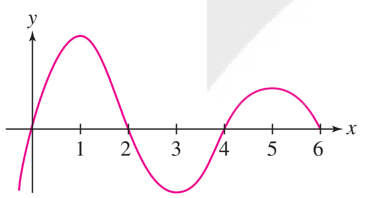


FIGURE 15

18. Figure 16 shows the graph of the derivative f' of a function f. Find the critical points of f and determine whether they are local minima, local maxima, or neither.

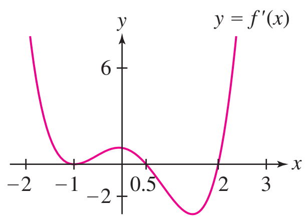


FIGURE 16

In Exercises 19–22, sketch the graph of a function f whose derivative f' has the given description.

19. $f'(x) > 0$ for $x > 3$ and $f'(x) < 0$ for $x < 3$

20. $f'(x) > 0$ for $x < 1$ and $f'(x) < 0$ for $x > 1$

21. $f'(x)$ is negative on $(1, 3)$ and positive everywhere else.

22. $f'(x)$ makes the sign transitions $+, -, +, -$.

In Exercises 23–26, find all critical points of f and use the First Derivative Test to determine whether they are local minima or local maxima.

23. $f(x) = 4 + 6x - x^2$

24. $f(x) = x^3 - 12x - 4$

25. $f(x) = \dfrac{x^2}{x+1}$

26. $f(x) = x^3 + x^{-3}$

In Exercises 27–58, find the critical points and the intervals on which the function is increasing or decreasing. Use the First Derivative Test to determine whether the critical point yields a local min or max (or neither).

27. $y = -x^2 + 7x - 17$

28. $y = 5x^2 + 6x - 4$

29. $y = x^3 - 12x^2$

30. $y = x(x-2)^3$

31. $y = 3x^4 + 8x^3 - 6x^2 - 24x$

32. $y = x^2 + (10 - x)^2$

33. $y = \frac{1}{3}x^3 + \frac{3}{2}x^2 + 2x + 4$

34. $y = x^4 + x^3$

35. $y = x^5 + x^3 + 1$

36. $y = x^5 + x^3 + x$

37. $y = x^4 - 4x^{3/2}$ $(x > 0)$

38. $y = x^{5/2} - x^2$ $(x > 0)$

39. $y = x + x^{-1}$

40. $y = x^{-2} - 4x^{-1}$

41. $y = \dfrac{1}{x^2+1}$

42. $y = \dfrac{2x+1}{x^2+1}$

43. $y = \dfrac{x^3}{x^2+1}$

44. $y = \dfrac{x^3}{x^2-3}$

45. $y = \theta + \sin\theta + \cos\theta$, $[0, 2\pi]$

46. $y = \sin\theta + \sqrt{3}\cos\theta$, $[0, 2\pi]$

47. $y = \sin^2\theta + \sin\theta$, $[0, 2\pi]$

48. $y = \theta - 2\cos\theta$, $[0, 2\pi]$

49. $y = x + e^{-x}$

50. $y = \dfrac{e^x}{x}$

51. $y = e^{-x}\cos x$, $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

52. $y = x^2 e^x$

53. $y = \tan^{-1} x - \frac{1}{2}x$

54. $y = (x^2 - 2x)e^x$

55. $y = x - \ln x^2$

56. $y = \dfrac{\ln x^2}{x}$

57. $y = x^{1/3}$

58. $y = x^{2/3} - x^2$

59. Find the maximum value of $f(x) = x^{-x}$ for $x > 0$.

60. Show that $f(x) = x^2 + bx + c$ is decreasing on $\left(-\infty, -\frac{b}{2}\right)$ and increasing on $\left(-\frac{b}{2}, \infty\right)$.

61. Show that $f(x) = x^3 - 2x^2 + 2x$ is an increasing function. *Hint:* Find the minimum value of f'.

62. Find conditions on a and b that ensure $f(x) = x^3 + ax + b$ is increasing on $(-\infty, \infty)$.

63. Ron's toll pass recorded him entering the tollway at mile 0 at 12:17 PM. He exited at mile 115 at 1:52 PM, and soon thereafter he was pulled over by the state police. "The speed limit on the tollway is 65 miles per hour," the trooper told Ron. "You exceeded that by more than five miles per hour this afternoon." "No way!" responded Ron. "I glance at the speedometer frequently, and not once did it read over 65!" How did the trooper use the Mean Value Theorem to support her claim that Ron must have gone more than 70 miles per hour at some point?

64. Two days after he bought a speedometer for his bicycle, Lance brought it back to the Yellow Jersey Bike Shop. "There is a problem with this speedometer," Lance complained to the clerk. "Yesterday I cycled the 22-mile Rogadzo Road Trail in 78 minutes, and not once did the speedometer read above 15 miles per hour!" "Yeah?" responded the clerk. "What's the problem?" How did Lance use the Mean Value Theorem to explain his complaint?

65. Determine where $f(x) = (1{,}000 - x)^2 + x^2$ is decreasing. Use this to decide which is larger: $800^2 + 200^2$ or $600^2 + 400^2$.

66. Show that $f(x) = 1 - |x|$ satisfies the conclusion of the MVT on $[a, b]$ if both a and b are positive or negative, but not if $a < 0$ and $b > 0$.

67. Which values of c satisfy the conclusion of the MVT on the interval $[a, b]$ if f is a linear function?

68. Show that if f is any quadratic polynomial, then the midpoint $c = \dfrac{a+b}{2}$ satisfies the conclusion of the MVT on $[a, b]$ for any a and b.

69. Suppose that $f(0) = 2$ and $f'(x) \le 3$ for $x > 0$. Apply the MVT to the interval $[0, 4]$ to prove that $f(4) \le 14$. Prove more generally that $f(x) \le 2 + 3x$ for all $x > 0$.

70. Show that if $f(2) = -2$ and $f'(x) \ge 5$ for $x > 2$, then $f(4) \ge 8$.

71. Show that if $f(2) = 5$ and $f'(x) \ge 10$ for $x > 2$, then $f(x) \ge 10x - 15$ for all $x > 2$.

Further Insights and Challenges

72. Show that a cubic function $f(x) = x^3 + ax^2 + bx + c$ is increasing on $(-\infty, \infty)$ if $b > a^2/3$.

73. Prove that if $f(0) = g(0)$ and $f'(x) \le g'(x)$ for $x \ge 0$, then $f(x) \le g(x)$ for all $x \ge 0$. *Hint:* Show that the function given by $y = f(x) - g(x)$ is nonincreasing.

74. Use Exercise 73 to prove that $x \le \tan x$ for $0 \le x < \frac{\pi}{2}$ and $\sin x \le x$ for $x \ge 0$.

75. Use Exercises 73 and 74 to prove the following assertions for all $x \ge 0$ (each assertion follows from the previous one):

(a) $\cos x \ge 1 - \frac{1}{2}x^2$

(b) $\sin x \ge x - \frac{1}{6}x^3$

(c) $\cos x \le 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$

Can you guess the next inequality in the series?

76. Let $f(x) = e^{-x}$. Use the method of Exercise 75 to prove the following inequalities for $x \ge 0$:

(a) $e^{-x} \ge 1 - x$

(b) $e^{-x} \le 1 - x + \frac{1}{2}x^2$

(c) $e^{-x} \ge 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3$

Can you guess the next inequality in the series?

77. Assume that f'' exists and $f''(x) = 0$ for all x. Prove that $f(x) = mx + b$, where $m = f'(0)$ and $b = f(0)$.

78. Define $f(x) = x^3 \sin\left(\frac{1}{x}\right)$ for $x \ne 0$ and $f(0) = 0$.

(a) Show that f' is continuous at $x = 0$ and that $x = 0$ is a critical point of f.

(b) GU Examine the graphs of f and f'. Can the First Derivative Test be applied?

(c) Show that $f(0)$ is neither a local min nor a local max.

79. Suppose that $f(x)$ satisfies the following equation (an example of a **differential equation**):

$$f''(x) = -f(x) \qquad \boxed{1}$$

(a) Show that $f(x)^2 + f'(x)^2 = f(0)^2 + f'(0)^2$ for all x. *Hint:* Show that the function on the left has zero derivative.

(b) Verify that $f(x) = \sin x$ and $f(x) = \cos x$ satisfy Eq. (1), and deduce that $\sin^2 x + \cos^2 x = 1$.

80. Suppose that functions f and g satisfy Eq. (1) and have the same initial values—that is, $f(0) = g(0)$ and $f'(0) = g'(0)$. Prove that $f(x) = g(x)$ for all x. *Hint:* Apply Exercise 79(a) to $f - g$.

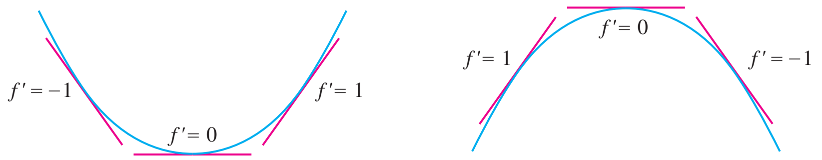
81. Use Exercise 80 to prove $f(x) = \sin x$ is the unique solution of Eq. (1) such that $f(0) = 0$ and $f'(0) = 1$; and $g(x) = \cos x$ is the unique solution such that $g(0) = 1$ and $g'(0) = 0$. This result can be used to develop all the properties of the trigonometric functions "analytically"—that is, without reference to triangles.

4.4 The Second Derivative and Concavity

In the previous section, we studied the increasing/decreasing behavior of a function, as determined by the sign of the derivative. Another important property is concavity, which refers to the way the graph bends. Informally, a curve is *concave up* if it bends up and *concave down* if it bends down (Figure 1).

FIGURE 1

To analyze concavity in a precise fashion, let's examine how concavity is related to tangent lines and derivatives. Observe in Figure 2 that when f is concave up, f' is increasing (the slopes of the tangent lines increase as we move to the right). Similarly, when f is concave down, f' is decreasing. This suggests the following definition.



Concave up: Slopes of tangent lines are increasing.

Concave down: Slopes of tangent lines are decreasing.

FIGURE 2

DEFINITION Concavity Let f be a differentiable function on an open interval (a, b). Then

- f is **concave up** on (a, b) if f' is increasing on (a, b).
- f is **concave down** on (a, b) if f' is decreasing on (a, b).

EXAMPLE 1 Concavity and Stock Prices The stocks of two companies, Arenot Industries (AI) and Blurbenthal Business Associates (BBA), went up in value, and both currently sell for \$75 (Figure 3). However, one is clearly a better investment than the other, assuming these trends continue in the same manner. Explain in terms of concavity.

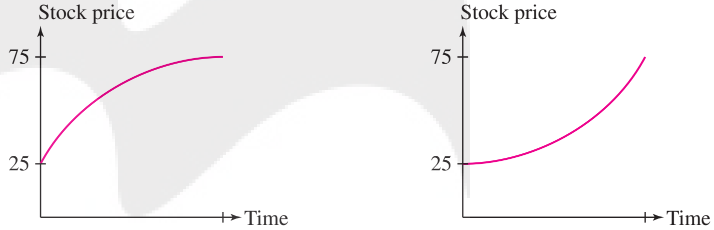


FIGURE 3

Solution The graph of Stock AI is concave down, so its growth rate (first derivative) is declining as time goes on. The graph of Stock BBA is concave up, so its growth rate is increasing. If these trends continue, Stock BBA is the better investment. ■

The concavity of a function is determined by the *sign* of its second derivative. Indeed, if $f''(x) > 0$, then f' is increasing and hence f is concave up. Similarly, if $f''(x) < 0$, then f' is decreasing and f is concave down.

THEOREM 1 Test for Concavity Assume that $f''(x)$ exists for all $x \in (a, b)$.

- If $f''(x) > 0$ for all $x \in (a, b)$, then f is concave up on (a, b).
- If $f''(x) < 0$ for all $x \in (a, b)$, then f is concave down on (a, b).

CAUTION A critical point c is just a single number, whereas a point of inflection $(c, f(c))$ is a point in the xy-plane.

Of special interest are the points on the graph where the concavity changes. We say that $P = (c, f(c))$ is a **point of inflection** of f if the concavity changes from up to down or from down to up at $x = c$. Figure 4 shows a curve made up of two arcs—one is concave down and one is concave up (the word "arc" refers to a piece of a curve). The point P where the arcs are joined is a point of inflection. We will denote points of inflection in graphs by a solid square ■.

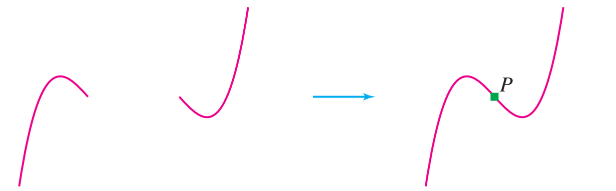


FIGURE 4

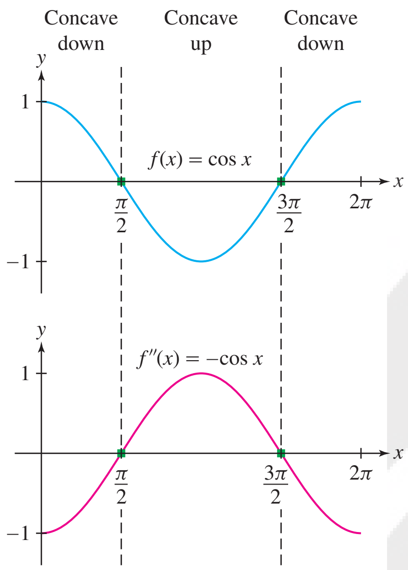


FIGURE 5

According to Theorem 1, the concavity of f is determined by the sign of $f''(x)$. Therefore, a point of inflection is a point where $f''(x)$ changes sign.

THEOREM 2 Test for Inflection Points If $f''(c) = 0$ or $f''(c)$ does not exist and $f''(x)$ changes sign at $x = c$, then f has a point of inflection at $x = c$.

EXAMPLE 2 Find the points of inflection of $f(x) = \cos x$ on $[0, 2\pi]$.

Solution We have

$$f''(x) = -\cos x, \quad \text{and} \quad f''(x) = 0 \quad \text{for } x = \frac{\pi}{2}, \frac{3\pi}{2}.$$

Figure 5 shows that $f''(x)$ changes sign at $x = \frac{\pi}{2}$ and $\frac{3\pi}{2}$, so f has a point of inflection at both points. ■

EXAMPLE 3 Points of Inflection and Intervals of Concavity Find the points of inflection and the intervals on which $f(x) = 3x^5 - 5x^4 + 1$ is concave up and concave down.

Solution The first derivative is $f'(x) = 15x^4 - 20x^3$ and

$$f''(x) = 60x^3 - 60x^2 = 60x^2(x - 1)$$

The zeros of $f''(x) = 60x^2(x - 1)$ are $x = 0$ and $x = 1$. They divide the x-axis into three intervals: $(-\infty, 0)$, $(0, 1)$, and $(1, \infty)$. We determine the sign of $f''(x)$ and the concavity of f by computing test values within each interval (Figure 6):

Interval	Test value	Sign of $f''(x)$	Behavior of $f(x)$
$(-\infty, 0)$	$f''(-1) = -120$	−	Concave down
$(0, 1)$	$f''\left(\frac{1}{2}\right) = -\frac{15}{2}$	−	Concave down
$(1, \infty)$	$f''(2) = 240$	+	Concave up

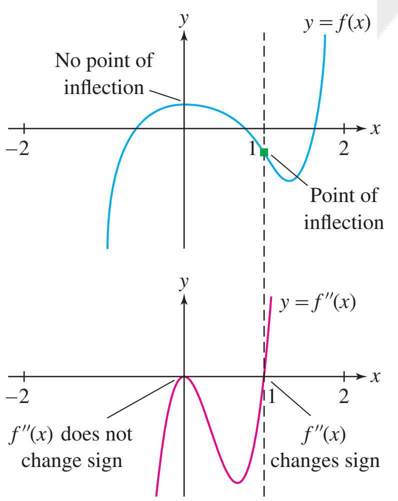


FIGURE 6 Graph of $f(x) = 3x^5 - 5x^4 + 1$ and its second derivative.

Since the concavity changes at $x = 1$ there is an inflection point there. The inflection point is $(1, -1)$. Note that, even though $f''(0) = 0$, there is not an inflection point at $x = 0$ because the concavity does not change at $x = 0$. ■

Usually, we find the inflection points by solving $f''(x) = 0$. However, an inflection point can also occur at a point $(c, f(c))$, where $f''(c)$ does not exist.

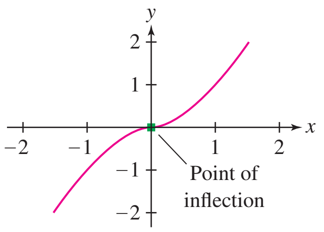


FIGURE 7 The concavity of $f(x) = x^{5/3}$ changes at $x = 0$ even though $f''(0)$ does not exist.

EXAMPLE 4 A Case Where the Second Derivative Does Not Exist Find the points of inflection of $f(x) = x^{5/3}$.

Solution In this case, $f'(x) = \frac{5}{3}x^{2/3}$ and $f''(x) = \frac{10}{9}x^{-1/3}$. Although $f''(0)$ does not exist, $f''(x)$ does change sign at $x = 0$:

$$f''(x) = \frac{10}{9x^{1/3}} \begin{cases} > 0 & \text{for } x > 0 \\ < 0 & \text{for } x < 0 \end{cases}$$

Therefore, the concavity of f changes at $x = 0$, and $(0, 0)$ is a point of inflection (Figure 7). ■

GRAPHICAL INSIGHT Points of inflection are easy to spot on the graph of the first derivative f'. If $f''(c) = 0$ and $f''(x)$ changes sign at $x = c$, then the increasing/decreasing behavior of f' changes at $x = c$:

- If $f''(x)$ goes from positive to negative at $x = c$, then f' has a local max at $x = c$.
- If $f''(x)$ goes from negative to positive at $x = c$, then f' has a local min at $x = c$.

Thus, *inflection points of f occur where f' has a local min or max* (Figure 8).

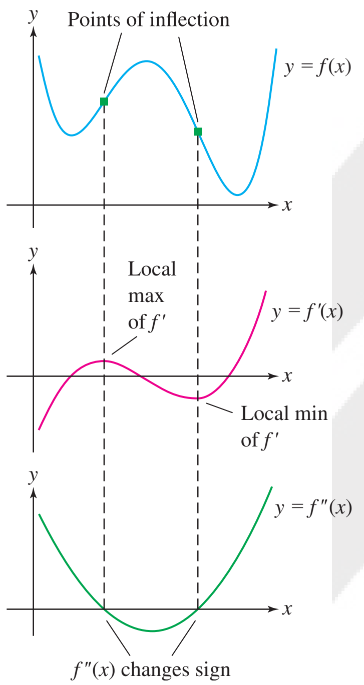


FIGURE 8

Second Derivative Test for Critical Points

There is a simple test for critical points based on concavity. Suppose that $f'(c) = 0$. As we see in Figure 9, $f(c)$ is a local max if f is concave down, and it is a local min if f is concave up. Concavity is determined by the sign of $f''(x)$, so we obtain the Second Derivative Test in Theorem 3. (See Exercise 73 for a detailed proof.)

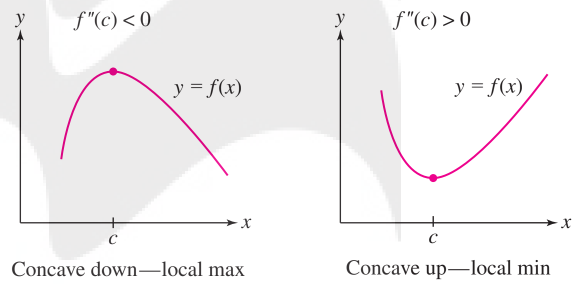


FIGURE 9 Concavity determines the type of the critical point.

THEOREM 3 Second Derivative Test Let c be a critical point of $f(x)$. If $f''(c)$ exists, then

- $f''(c) > 0 \quad \Rightarrow \quad f(c)$ is a local minimum.
- $f''(c) < 0 \quad \Rightarrow \quad f(c)$ is a local maximum.
- $f''(c) = 0 \quad \Rightarrow \quad$ inconclusive: $f(c)$ may be a local min, a local max, or neither.

Mnemonic Device:

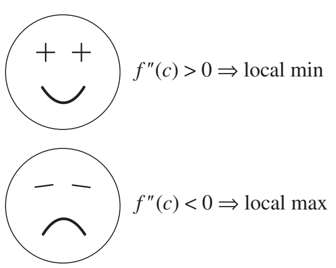


FIGURE 10

The mnemonic device appearing in Figure 10 provides an easy way to remember the test.

EXAMPLE 5 Analyze the critical points of $f(x) = (2x - x^2)e^x$.

Solution First, we have

$$f'(x) = e^x(2 - 2x) + (2x - x^2)e^x = (2 - x^2)e^x$$

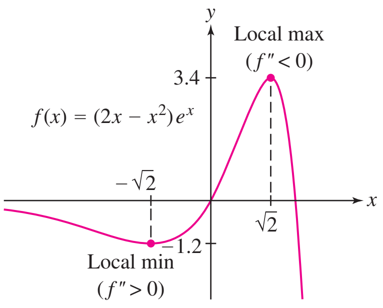


FIGURE 11

To find the critical points, solve $(2 - x^2)e^x = 0$. Therefore $c = \pm\sqrt{2}$. Next, determine the sign of the second derivative at the critical points:

$$f''(x) = (-2x)e^x + (2 - x^2)e^x = (2 - 2x - x^2)e^x$$

$$f''(-\sqrt{2}) = \left(2 - 2(-\sqrt{2}) - (-\sqrt{2})^2\right)e^{-\sqrt{2}} = 2\sqrt{2}e^{-\sqrt{2}} \quad > 0 \quad \text{(local min)}$$

$$f''(\sqrt{2}) = \left(2 - 2\sqrt{2} - (\sqrt{2})^2\right)e^{\sqrt{2}} = -2\sqrt{2}e^{\sqrt{2}} \quad < 0 \quad \text{(local max)}$$

By the Second Derivative Test, f has a local min at $x = -\sqrt{2}$ and a local max at $x = \sqrt{2}$ (Figure 11). ■

EXAMPLE 6 Second Derivative Test Inconclusive Analyze the critical points of $f(x) = x^5 - 5x^4$.

Solution The first two derivatives are

$$f'(x) = 5x^4 - 20x^3 = 5x^3(x - 4)$$

$$f''(x) = 20x^3 - 60x^2$$

The critical points are $c = 0, 4$, and the Second Derivative Test yields

$$f''(0) = 0 \quad \Rightarrow \quad \text{Second Derivative Test fails}$$

$$f''(4) = 320 > 0 \quad \Rightarrow \quad f(4) \text{ is a local min}$$

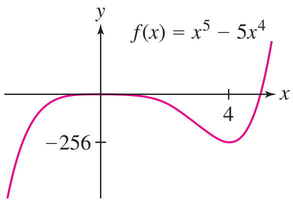


FIGURE 12

The Second Derivative Test fails at $x = 0$, so we fall back on the First Derivative Test. Choosing test points to the left and right of $x = 0$, we find

$$f'(-1) = 5 + 20 = 25 > 0 \quad \Rightarrow \quad f'(x) \text{ is positive on } (-\infty, 0)$$

$$f'(1) = 5 - 20 = -15 < 0 \quad \Rightarrow \quad f'(x) \text{ is negative on } (0, 4)$$

Since $f'(x)$ changes from $+$ to $-$ at $x = 0$, $f(0)$ is a local max (Figure 12). ■

4.4 SUMMARY

- A differentiable function f is *concave up* on (a, b) if f' is increasing and *concave down* if f' is decreasing on (a, b).
- The signs of the first two derivatives provide the following information:

First derivative			**Second derivative**		
$f' > 0$	$\Rightarrow$	f is increasing	$f'' > 0$	$\Rightarrow$	f is concave up
$f' < 0$	$\Rightarrow$	f is decreasing	$f'' < 0$	$\Rightarrow$	f is concave down

- A *point of inflection* is a point $(c, f(c))$ where the concavity changes from concave up to concave down, or vice versa.
- Second Derivative Test: If $f'(c) = 0$ and $f''(c)$ exists, then
  - $f(c)$ is a local maximum value if $f''(c) < 0$
  - $f(c)$ is a local minimum value if $f''(c) > 0$
  - The test fails if $f''(c) = 0$

  If this test fails, use the First Derivative Test.

4.4 EXERCISES

Preliminary Questions

1. If f is concave up, then f' is (choose one)

(a) increasing **(b)** decreasing

2. What conclusion can you draw if $f'(c) = 0$ and $f''(c) < 0$?

3. True or false? If $f(c)$ is a local min, then $f''(c)$ must be positive.

4. True or false? If $f''(c) = 0$, then f has an inflection point at $x = c$.

5. The function $f(x) = \frac{x^2+1}{x}$ is concave down for $x < 0$ and concave up for $x > 0$. Is there an inflection point at $x = 0$? Explain.

6. Can a function have an inflection point at a critical point? Explain.

Exercises

1. Match the graphs in Figure 13 with the description:

(a) $f''(x) < 0$ for all x. **(b)** $f''(x)$ goes from $+$ to $-$.

(c) $f''(x) > 0$ for all x. **(d)** $f''(x)$ goes from $-$ to $+$.

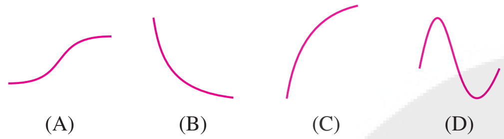


FIGURE 13

2. Match each statement with a graph in Figure 14 that represents company profits as a function of time.

(a) The outlook is great: The growth rate keeps increasing.

(b) We're losing money, but not as quickly as before.

(c) We're losing money, and it's getting worse as time goes on.

(d) We're doing well, but our growth rate is leveling off.

(e) Business had been cooling off, but now it's picking up.

(f) Business had been picking up, but now it's cooling off.

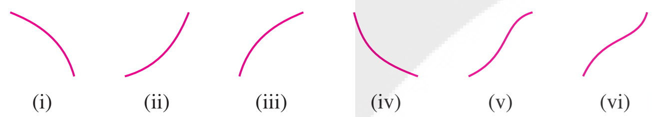


FIGURE 14

3. GU Plot $f(x) = (2x - x^2)e^x$ and indicate on the graph where it appears that inflection points occur. Then find the inflection points using calculus.

4. GU Plot $f(x) = x(x-4)^3$ and indicate on the graph where it appears that inflection points occur. Then find the inflection points using calculus.

In Exercises 5–24, determine the intervals on which the function is concave up or down and find the points of inflection.

5. $y = x^2 - 4x + 3$

6. $y = t^3 - 6t^2 + 4$

7. $y = 10x^3 - x^5$

8. $y = 5x^2 + x^4$

9. $y = \theta - 2\sin\theta, \quad [0, 2\pi]$

10. $y = \theta + \sin^2\theta, \quad [0, \pi]$

11. $y = x(x - 8\sqrt{x}) \quad (x \ge 0)$

12. $y = x^{7/2} - 35x^2$

13. $y = (x-2)(1-x^3)$

14. $y = x^{7/5}$

15. $y = \dfrac{1}{x^2+3}$

16. $y = \dfrac{x}{x^2+9}$

17. $f(x) = \dfrac{x^3}{1+x}$

18. $w(t) = \dfrac{t^4 - 1}{t}$

19. $y = xe^{-3x}$

20. $y = (x^2 - 7)e^x$

21. $y = 2x^2 + \ln x \quad (x > 0)$

22. $y = x - \ln x \quad (x > 0)$

23. $f(t) = te^{-t^2}$

24. The Surge Function $S(t) = Ate^{-kt}$, with $A, k > 0$

25. The position of an ambulance in kilometers on a straight road over a period of 4 hours is given by the graph in Figure 15.

(a) Describe the motion of the ambulance.

(b) Explain what the fact that this graph is concave up tells us about the speed of the ambulance.

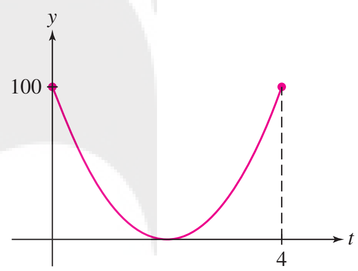


FIGURE 15

26. The position of a bicyclist on a straight road in kilometers over a period of 4 h is given by the graph in Figure 16, where inflection points occur when $t = 0.5$ and $t = 2$.

(a) Describe the motion of the bicyclist.

(b) Explain what the concavity of the graph over various intervals tells us about the speed of the bicyclist.

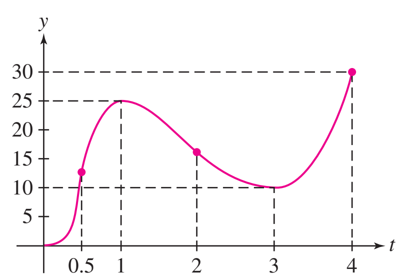


FIGURE 16

27. The growth of a sunflower during the first 100 days after sprouting is modeled well by the *logistic curve* $y = h(t)$ shown in Figure 17. Estimate the growth rate at the point of inflection and explain its significance. Then make a rough sketch of the first and second derivatives of h.

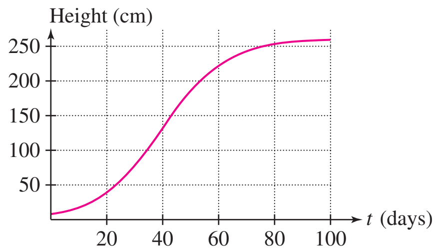


FIGURE 17

28. Assume that Figure 18 is the graph of f. Where do the points of inflection of f occur, and on which interval is f concave down?

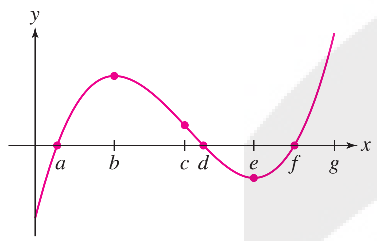


FIGURE 18

29. Repeat Exercise 28 but assume that Figure 18 is the graph of the *derivative* f'.

30. Repeat Exercise 28 but assume that Figure 18 is the graph of the *second derivative* f''.

31. Figure 19 shows the *derivative* f' on $[0, 1.2]$. Locate the points of inflection of f and the points where the local minima and maxima occur. Determine the intervals on which f has the following properties:

(a) Increasing **(b)** Decreasing

(c) Concave up **(d)** Concave down

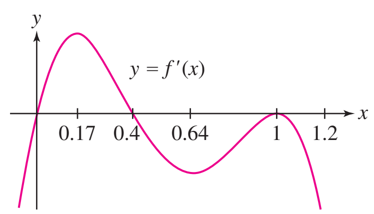


FIGURE 19

32. Leticia has been selling solar-powered laptop chargers through her Web site, with monthly sales as recorded below. In a report to investors, she states, "Sales reached a point of inflection when I started using pay-per-click advertising." In which month did that occur? Explain.

Month	1	2	3	4	5	6	7	8
Sales	2	30	50	60	90	150	230	340

In Exercises 33–46, find the critical points and apply the Second Derivative Test (or state that it fails).

33. $f(x) = x^3 - 12x^2 + 45x$

34. $f(x) = x^4 - 8x^2 + 1$

35. $f(x) = 3x^4 - 8x^3 + 6x^2$

36. $f(x) = x^5 - x^3$

37. $f(x) = \dfrac{x^2 - 8x}{x + 1}$

38. $f(x) = \dfrac{1}{x^2 - x + 2}$

39. $y = 6x^{3/2} - 4x^{1/2}$

40. $y = 9x^{7/3} - 21x^{1/2}$

41. $f(x) = \sin^2 x + \cos x, \quad [0, \pi]$

42. $y = \dfrac{1}{\sin x + 4}, \quad [0, 2\pi]$

43. $f(x) = xe^{-x^2}$

44. $f(x) = e^{-x} - 4e^{-2x}$

45. $f(x) = x^3 \ln x \quad (x > 0)$

46. $f(x) = \ln x + \ln(4 - x^2), \quad (0, 2)$

In Exercises 47–62, find the intervals on which f is concave up or down, the points of inflection, the critical points, and the local minima and maxima.

47. $f(x) = x^3 - 2x^2 + x$

48. $f(x) = x^2(x - 4)$

49. $f(t) = t^2 - t^3$

50. $f(x) = 2x^4 - 3x^2 + 2$

51. $f(x) = x^2 - 8x^{1/2} \quad (x \geq 0)$

52. $f(x) = x^{3/2} - 4x^{-1/2} \quad (x > 0)$

53. $f(x) = \dfrac{x}{x^2 + 27}$

54. $f(x) = \dfrac{1}{x^4 + 1}$

55. $f(x) = x^{5/3} - x$

56. $f(x) = (x - 1)^{3/5}$

57. $f(\theta) = \theta + \sin\theta, \quad [0, 2\pi]$

58. $f(x) = \cos^2 x, \quad [0, \pi]$

59. $f(x) = \tan x, \quad \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

60. $f(x) = e^{-x}\cos x, \quad \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$

61. $y = (x^2 - 2)e^{-x} \quad (x > 0)$

62. $y = \ln(x^2 + 2x + 5)$

63. Sketch the graph of an increasing function such that $f''(x)$ changes from $+$ to $-$ at $x = 2$ and from $-$ to $+$ at $x = 4$. Do the same for a decreasing function.

In Exercises 64–66, sketch the graph of a function f satisfying all of the given conditions.

64. $f'(x) > 0$ and $f''(x) < 0$ for all x

65. **(i)** $f'(x) > 0$ for all x, and

(ii) $f''(x) < 0$ for $x < 0$ and $f''(x) > 0$ for $x > 0$

66. **(i)** $f'(x) < 0$ for $x < 0$ and $f'(x) > 0$ for $x > 0$, and

(ii) $f''(x) < 0$ for $|x| > 2$, and $f''(x) > 0$ for $|x| < 2$

67. An infectious flu spreads slowly at the beginning of an epidemic. The infection process accelerates until a majority of the susceptible individuals are infected, at which point the process slows down.

(a) If $R(t)$ is the number of individuals infected at time t, describe the concavity of the graph of R near the beginning and end of the epidemic.

(b) Describe the status of the epidemic on the day that R has a point of inflection.

68. Water is pumped into a sphere at a constant rate (Figure 20). Let $h(t)$ be the water level at time t. Sketch the graph of h (approximately, but with the correct concavity). Where does the point of inflection occur?

69. Water is pumped into a sphere of radius R at a variable rate in such a way that the water level rises at a constant rate (Figure 20). Let $V(t)$ be the volume of water in the tank at time t. Sketch the graph V (approximately, but with the correct concavity). Where does the point of inflection occur?

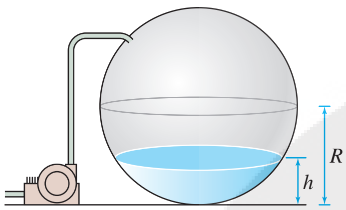


FIGURE 20

70. (Continuation of Exercise 69) If the sphere has radius R, the volume of water is

$$V = \pi\left(Rh^2 - \frac{1}{3}h^3\right),$$

where h is the water level. Assume the level rises at a constant rate of 1 (i.e., $h = t$).

(a) Find the inflection point of V. Does this agree with your conclusion in Exercise 69?

(b) GU Plot V for $R = 1$.

71. Image Processing The intensity of a pixel in a digital image is measured by a number u between 0 and 1. Often, images can be enhanced by rescaling intensities, as in the images of Amelia Earhart in Figure 21. When rescaling, pixels of intensity u are displayed with intensity $g(u)$ for a suitable function g. One common choice is the **sigmoidal correction**, defined for constants a, b by

$$g(u) = \frac{f(u) - f(0)}{f(1) - f(0)}, \qquad \text{where} \quad f(u) = \left(1 + e^{b(a-u)}\right)^{-1}$$

Original Sigmoidal correction

Both: Library of Congress Prints and Photographs Division

FIGURE 21

Figure 22 shows that $g(u)$ reduces the intensity of low-intensity pixels [where $g(u) < u$] and increases the intensity of high-intensity pixels.

(a) Verify that $f'(u) > 0$ and use this to show that $g(u)$ increases from 0 to 1 for $0 \le u \le 1$.

(b) Where does $g(u)$ have a point of inflection?

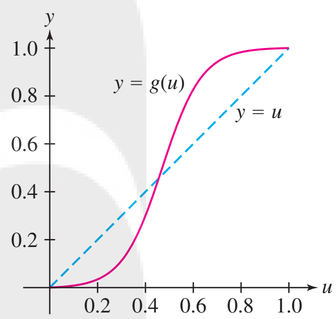


FIGURE 22 Sigmoidal correction with $a = 0.47$, $b = 12$.

72. Use graphical reasoning to determine whether the following statements are true or false. If false, modify the statement to make it correct.

(a) If f is increasing, then f^{-1} is decreasing.

(b) If f is decreasing, then f^{-1} is decreasing.

(c) If f is concave up, then f^{-1} is concave up.

(d) If f is concave down, then f^{-1} is concave up.

Further Insights and Challenges

In Exercises 73–75, assume that f is differentiable.

73. Proof of the Second Derivative Test Let c be a critical point such that $f''(c) > 0$ [the case $f''(c) < 0$ is similar].

(a) Show that $f''(c) = \lim\limits_{h\to 0} \dfrac{f'(c+h)}{h}$.

(b) Use (a) to show that there exists an open interval (a, b) containing c such that $f'(x) < 0$ if $a < x < c$ and $f'(x) > 0$ if $c < x < b$. Conclude that $f(c)$ is a local minimum.

74. Prove that if f'' exists and $f''(x) > 0$ for all x, then the graph of f "sits above" its tangent lines.

(a) For any c, set $G(x) = f(x) - f'(c)(x - c) - f(c)$. It is sufficient to prove that $G(x) \ge 0$ for all c. Explain why with a sketch.

(b) Show that $G(c) = G'(c) = 0$ and $G''(x) > 0$ for all x. Conclude that $G'(x) < 0$ for $x < c$ and $G'(x) > 0$ for $x > c$. Then deduce, using the MVT, that $G(x) > G(c)$ for $x \ne c$.

75. Assume that f'' exists and let c be a point of inflection of f.

(a) Use the method of Exercise 74 to prove that the tangent line at $x = c$ *crosses the graph* (Figure 23). *Hint:* Show that $G(x)$ changes sign at $x = c$.

(b) GU Verify this conclusion for $f(x) = \dfrac{x}{3x^2 + 1}$ by graphing f and the tangent line at each inflection point on the same set of axes.

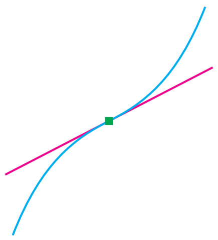


FIGURE 23 Tangent line crosses graph at point of inflection.

76. Let $C(x)$ be the cost of producing x units of a certain good. Assume that the graph of C is concave up.

(a) Show that the average cost $A(x) = C(x)/x$ is minimized at the production level x_0 such that average cost equals marginal cost—that is, $A(x_0) = C'(x_0)$.

(b) Show that the line through $(0, 0)$ and $(x_0, C(x_0))$ is tangent to the graph of C.

77. Let f be a polynomial of degree $n \geq 2$. Show that f has at least one point of inflection if n is odd. Then give an example to show that f need not have a point of inflection if n is even.

78. Critical and Inflection Points If $f'(c) = 0$ and $f(c)$ is neither a local min nor a local max, must $x = c$ be a point of inflection? This is true for "reasonable" functions (including the functions studied in this text), but it is not true in general. Let

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

(a) Use the limit definition of the derivative to show that $f'(0)$ exists and $f'(0) = 0$.

(b) Show that $f(0)$ is neither a local min nor a local max.

(c) Show that $f'(x)$ changes sign infinitely often near $x = 0$. Conclude that $x = 0$ is not a point of inflection.

4.5 L'Hôpital's Rule

L'Hôpital's Rule is named for the French mathematician Guillaume François Antoine Marquis de L'Hôpital (1661–1704), who wrote the first textbook on calculus in 1696. The name L'Hôpital is pronounced "Lo-pee-tal."

L'Hôpital's Rule is a valuable tool for computing certain limits that are otherwise difficult to evaluate, and also for determining "asymptotic behavior" (limits at infinity). We will use it for graph sketching in the next section.

Consider the limit of a quotient:

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

Roughly speaking, L'Hôpital's Rule states that *when $f(x)/g(x)$ has an indeterminate form of type $0/0$ or ∞/∞ at $x = a$, then we can replace $f(x)/g(x)$ by the quotient of the derivatives $f'(x)/g'(x)$.*

THEOREM 1 L'Hôpital's Rule Assume that f and g are differentiable on an open interval containing a and that

$$f(a) = g(a) = 0$$

Also assume that $g'(x) \neq 0$ (except possibly at a). Then

$$\boxed{\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}}$$

if the limit on the right exists or is infinite (∞ or $-\infty$). This conclusion also holds if f and g are differentiable for x near (but not equal to) a and

$$\lim_{x \to a} f(x) = \pm\infty \qquad \text{and} \qquad \lim_{x \to a} g(x) = \pm\infty$$

Furthermore, this rule is valid for one-sided limits.

CAUTION *When using L'Hôpital's Rule, be sure to take the derivative of the numerator and denominator separately:*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Do not take the derivative of the function $y = f(x)/g(x)$ as a quotient, for example using the Quotient Rule.

EXAMPLE 1 Use L'Hôpital's Rule to evaluate $\lim\limits_{x \to 2} \dfrac{x^3 - 8}{x^4 + 2x - 20}$.

Solution Let $f(x) = x^3 - 8$ and $g(x) = x^4 + 2x - 20$. Both f and g are differentiable and $f(x)/g(x)$ is indeterminate of type $0/0$ at $a = 2$ because $f(2) = g(2) = 0$.

Furthermore, $g'(x) = 4x^3 + 2$ is nonzero near $x = 2$, so L'Hôpital's Rule applies. We may replace the numerator and denominator by their derivatives to obtain

$$\underbrace{\lim_{x\to 2} \frac{x^3 - 8}{x^4 + 2x - 2} = \lim_{x\to 2} \frac{(x^3 - 8)'}{(x^4 + 2x - 2)'}}_{\text{L'Hôpital's Rule}} = \lim_{x\to 2} \frac{3x^2}{4x^3 + 2} = \frac{3(2^2)}{4(2^3) + 2} = \frac{12}{34} = \frac{6}{17} \quad \blacksquare$$

EXAMPLE 2 Evaluate $\displaystyle\lim_{x\to\pi/2} \frac{\cos^2 x}{1 - \sin x}$.

Solution Again, the quotient is indeterminate of type $0/0$ at $x = \frac{\pi}{2}$ since

$$\cos^2\left(\frac{\pi}{2}\right) = 0, \qquad 1 - \sin\frac{\pi}{2} = 1 - 1 = 0$$

The other hypotheses are satisfied, so we may apply L'Hôpital's Rule:

$$\underbrace{\lim_{x\to\pi/2} \frac{\cos^2 x}{1 - \sin x} = \lim_{x\to\pi/2} \frac{(\cos^2 x)'}{(1 - \sin x)'}}_{\text{L'Hôpital's Rule}} = \lim_{x\to\pi/2} \frac{-2\cos x\ \sin x}{-\cos x} = \underbrace{\lim_{x\to\pi/2} (2\sin x)}_{\text{Simplified}} = 2$$

Note that the quotient $\dfrac{-2\cos x\ \sin x}{-\cos x}$ is also indeterminate at $x = \pi/2$. We removed this indeterminacy by cancelling the factor $-\cos x$. ■

EXAMPLE 3 **The Form $0 \cdot \infty$** Evaluate $\displaystyle\lim_{x\to 0^+} x \ln x$.

Solution This limit is one-sided because $f(x) = x \ln x$ is not defined for $x \leq 0$. Furthermore, as $x \to 0^+$,

- x approaches 0.
- $\ln x$ approaches $-\infty$.

So $f(x)$ presents an indeterminate form of type $0 \cdot \infty$. To apply L'Hôpital's Rule, we rewrite our function as $f(x) = (\ln x)/x^{-1}$ so that $f(x)$ presents an indeterminate form of type $-\infty/\infty$. Then L'Hôpital's Rule applies:

$$\lim_{x\to 0^+} x \ln x = \underbrace{\lim_{x\to 0^+} \frac{\ln x}{x^{-1}} = \lim_{x\to 0^+} \frac{(\ln x)'}{(x^{-1})'}}_{\text{L'Hôpital's Rule}} = \lim_{x\to 0^+} \left(\frac{x^{-1}}{-x^{-2}}\right) = \underbrace{\lim_{x\to 0^+} (-x)}_{\text{Simplified}} = 0 \quad \blacksquare$$

EXAMPLE 4 **Using L'Hôpital's Rule Twice** Evaluate $\displaystyle\lim_{x\to 0} \frac{e^x - x - 1}{\cos x - 1}$.

Solution The limit is in the indeterminate form $0/0$ since at $x = 0$, we have

$$e^x - x - 1 = e^0 - 0 - 1 = 0, \qquad \cos x - 1 = \cos 0 - 1 = 0$$

A first application of L'Hôpital's Rule gives

$$\lim_{x\to 0} \frac{e^x - x - 1}{\cos x - 1} = \lim_{x\to 0} \frac{(e^x - x - 1)'}{(\cos x - 1)'} = \lim_{x\to 0} \left(\frac{e^x - 1}{-\sin x}\right) = \lim_{x\to 0} \frac{1 - e^x}{\sin x}$$

This limit is again indeterminate of type $0/0$, so we apply L'Hôpital's Rule a second time:

$$\lim_{x\to 0} \frac{1 - e^x}{\sin x} = \lim_{x\to 0} \frac{-e^x}{\cos x} = \frac{-e^0}{\cos 0} = -1$$

It follows that

$$\lim_{x\to 0} \frac{e^x - x - 1}{\cos x - 1} = -1 \quad \blacksquare$$

EXAMPLE 5 Maximum Height Under Air Resistance In Example 7 in Section 2.5 we introduced a function

$$H(k) = \frac{30k - 9.8\ln\left(\frac{150k}{49} + 1\right)}{k^2}$$

that gives the maximum height attained by a one kilogram ball launched upward at 30 m/s with gravity and air resistance acting on it. (The function is derived in Section 9.2.) The variable k reflects the strength of the air resistance. We investigated what happens to the maximum height as the air resistance approaches zero; that is, we investigated $\lim_{k\to 0} H(k)$ numerically. Show this limit can be evaluated using L'Hôpital's Rule and find the limit.

Solution The quotient $\frac{30k-9.8\ln\left(\frac{150k}{49}+1\right)}{k^2}$ has the indeterminate form 0/0. To evaluate the limit, we need to use L'Hôpital's Rule twice:

$$\lim_{k\to 0} \frac{30k - 9.8\ln\left(\frac{150k}{49} + 1\right)}{k^2} = \lim_{k\to 0} \frac{30 - \frac{30}{\frac{150k}{49}+1}}{2k} = \lim_{k\to 0} \frac{\frac{4500/49}{\left(\frac{150k}{49}+1\right)^2}}{2} = \frac{2250}{49} \approx 45.92$$

This value of 45.92 m matches our previous numerical estimate and the result we obtained separately in Example 7 in Section 3.4 where we considered the launched projectile's height, ignoring air resistance altogether.

EXAMPLE 6 Assumptions Matter Can L'Hôpital's Rule be applied to $\lim_{x\to 1} \frac{x^2+1}{2x+1}$?

Solution The answer is no. The function does *not* have an indeterminate form because

$$\left.\frac{x^2+1}{2x+1}\right|_{x=1} = \frac{1^2+1}{2\cdot 1+1} = \frac{2}{3}$$

This limit can be evaluated directly by substitution: $\lim_{x\to 1} \frac{x^2+1}{2x+1} = \frac{2}{3}$. An incorrect application of L'Hôpital's Rule gives the wrong answer:

$$\lim_{x\to 1} \frac{(x^2+1)'}{(2x+1)'} = \lim_{x\to 1} \frac{2x}{2} = 1 \quad \text{(not equal to original limit)}$$

EXAMPLE 7 The Form $\infty - \infty$ Evaluate $\lim_{x\to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right)$.

Solution Both $1/\sin x$ and $1/x$ become infinite at $x = 0$, so we have an indeterminate form of type $\infty - \infty$. We rewrite the function as

$$\frac{1}{\sin x} - \frac{1}{x} = \frac{x - \sin x}{x \sin x}$$

to obtain an indeterminate form of type $0/0$. Applying L'Hôpital's Rule twice yields

$$\lim_{x\to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right) = \underbrace{\lim_{x\to 0} \frac{x - \sin x}{x\sin x} = \lim_{x\to 0} \frac{1-\cos x}{x\cos x + \sin x}}_{\text{L'Hôpital's Rule}}$$

$$= \underbrace{\lim_{x\to 0} \frac{\sin x}{-x\sin x + 2\cos x}}_{\text{L'Hôpital's Rule again}} = \frac{0}{2} = 0$$

This value of the limit is confirmed graphically in Figure 1.

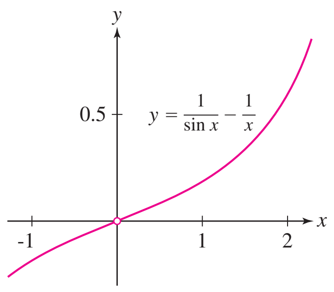


FIGURE 1 The graph confirms that $y = \frac{1}{\sin x} - \frac{1}{x}$ approaches 0 as $x \to 0$.

Limits of functions of the form $f(x)^{g(x)}$ can lead to the indeterminate forms 0^0, 1^∞, or ∞^0. These are indeterminate since the limit can take on a variety of values, depending on the relative rates at which the base and exponent approach their limits. In evaluating these limits, we use the change-of-base formula to write $f(x)^{g(x)} = e^{g(x)\ln f(x)}$ and then we obtain

$$\lim_{x\to a} f(x)^{g(x)} = \lim_{x\to a} e^{g(x)\ln f(x)} = e^{\lim_{x\to a} g(x)\ln f(x)}$$

The last equality is justified by the continuity of the exponential function.

REMINDER The change-of-base formula, changing an exponential base a to base e, is $a^r = e^{r\ln a}$.

EXAMPLE 8 The Form 0^0 Evaluate $\lim_{x\to 0+} x^x$.

Solution With $x^x = e^{x\ln x}$ by the change-of-base formula, it will be enough to consider the limit of $x\ln x$. Example 3 showed $\lim_{x\to 0+} x\ln x = 0$. Therefore,

$$\lim_{x\to 0+} x^x = \lim_{x\to 0+} e^{x\ln x} = e^{\lim_{x\to 0+} x\ln x} = e^0 = 1$$

This value for the limit is confirmed graphically in Figure 2. ■

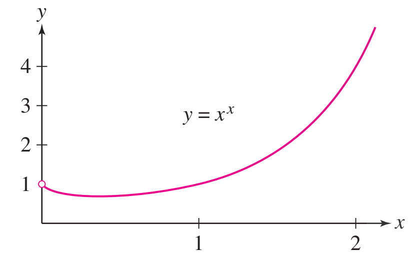


FIGURE 2 The function $y = x^x$ approaches 1 as $x \to 0^+$.

In Section 1.6, we pointed out that e is the value that $(1+x)^{1/x}$ approaches as x approaches 0. This can be verified now by evaluating $\lim_{x\to 0}(1+x)^{1/x}$ using L'Hôpital's Rule.

EXAMPLE 9 The Form 1^∞ Find $\lim_{x\to 0}(1+x)^{1/x}$.

Solution This has the indeterminate form 1^∞. We take the approach used in Example 8. Thus, we write $(1+x)^{1/x} = e^{\frac{1}{x}\ln(1+x)}$ and consider $\lim_{x\to 0}\frac{1}{x}\ln(1+x)$. We obtain (using L'Hôpital's Rule for the first equality)

$$\lim_{x\to 0}\frac{\ln(1+x)}{x} = \lim_{x\to 0}\frac{\frac{1}{1+x}}{1} = 1$$

Therefore,

$$\lim_{x\to 0}(1+x)^{1/x} = \lim_{x\to 0} e^{\frac{1}{x}\ln(1+x)} = e^{\lim_{x\to 0}\frac{\ln(1+x)}{x}} = e^1 = e$$ ■

Note that if we substitute $x = \frac{1}{t}$ into $\lim_{t\to\infty}\left(1+\frac{1}{t}\right)^t$ we obtain the limit in the previous example. Therefore $\lim_{t\to\infty}\left(1+\frac{1}{t}\right)^t = e$. It is important to be familiar with these limits whose values are e:

$$e = \lim_{x\to 0}(1+x)^{1/x} \quad \text{and} \quad e = \lim_{t\to\infty}\left(1+\frac{1}{t}\right)^t$$

They arise in limit evaluations that we will see subsequently in the text.

CONCEPTUAL INSIGHT Exponential Limit Forms Knowing that $0\cdot\infty$ is an indeterminate form, and using the exponential identity $a^x = e^{x\ln a}$, we can see why 0^0, 1^∞, and ∞^0 are indeterminate forms. A similar approach also shows why 0^∞ is not indeterminate and corresponds to a limit that equals 0.

The Form 0^0: If $\lim_{x\to a} f(x)^{g(x)}$ is in the form 0^0, then $f(x)\to 0$ and $g(x)\to 0$. Therefore, in the limit, the equivalent exponential expression $e^{g(x)\ln f(x)}$ has an exponent in the indeterminate form $0(-\infty)$ since $g(x)\to 0$ and $\ln f(x)\to -\infty$ (because $f(x)\to 0$). Therefore, 0^0 is an indeterminate form.

Similar arguments can be made to demonstrate that 1^∞ and ∞^0 are indeterminate forms (see Exercise 61).

The Form 0^∞: If $\lim_{x\to a} f(x)^{g(x)}$ is in the form 0^∞, then $f(x)\to 0$ and $g(x)\to\infty$. Therefore, in the limit, the equivalent exponential expression $e^{g(x)\ln f(x)}$ has an exponent in the form $(\infty)(-\infty)$. Since the limit of the exponent is $-\infty$, it follows that the limit of $e^{g(x)\ln f(x)}$ is 0, and therefore the limit of $f(x)^{g(x)}$ is as well. Thus, the form 0^∞ is not indeterminate but instead corresponds to a limit that is equal to 0.

Comparing Growth of Functions

Sometimes, we are interested in determining which of two given functions grows faster. For example, Quick Sort and Bubble Sort are two standard computer algorithms for sorting data (e.g., alphabetizing, ordering according to rank). The average time required to sort a list of size n is approxiamtely $n\ \ln\ n$ for Quick Sort and n^2 for Bubble Sort. Which algorithm is faster when the size n is large? This problem amounts to comparing the growth of $Q(x) = x\ln x$ and $B(x) = x^2$ as $x\to\infty$.

We say that $f(x)$ grows *faster* than $g(x)$ if

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \infty \qquad \text{or, equivalently,} \qquad \lim_{x\to\infty}\frac{g(x)}{f(x)} = 0$$

To indicate that $f(x)$ grows faster than $g(x)$, we use the notation $g(x) \ll f(x)$. For example, $x \ll x^2$ because

$$\lim_{x\to\infty}\frac{x^2}{x} = \lim_{x\to\infty} x = \infty$$

To compare the growth of functions, we need a version of L'Hôpital's Rule that applies to limits at infinity.

THEOREM 2 L'Hôpital's Rule for Limits at Infinity Assume that f and g are differentiable in an interval (b,∞) and that $g'(x)\neq 0$ for $x > b$. If $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} g(x)$ exist and either both are zero or both are infinite, then

$$\boxed{\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{f'(x)}{g'(x)}}$$

provided that the limit on the right exists. A similar result holds for limits as $x\to-\infty$.

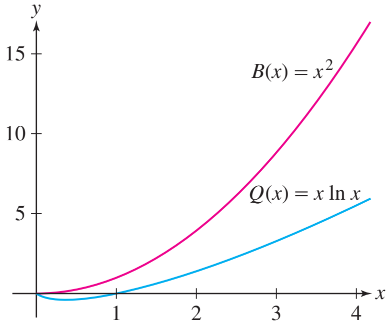


FIGURE 3

EXAMPLE 10 The Form $\frac{\infty}{\infty}$ Which of $B(x) = x^2$ or $Q(x) = x\ln x$ grows faster as $x\to\infty$?

Solution Both $B(x)$ and $Q(x)$ approach infinity as $x\to\infty$, so L'Hôpital's Rule applies to the quotient:

$$\lim_{x\to\infty}\frac{B(x)}{Q(x)} = \lim_{x\to\infty}\frac{x^2}{x\ln x} = \underbrace{\lim_{x\to\infty}\frac{x}{\ln x} = \lim_{x\to\infty}\frac{1}{x^{-1}}}_{\text{L'Hôpital's Rule}} = \lim_{x\to\infty} x = \infty$$

We conclude that $x\ln x \ll x^2$ (Figure 3). ■

Note that this example implies that Quick Sort is a much faster sorting algorithm than Bubble Sort for large n.

In Section 1.6, we asserted that exponential functions increase more rapidly than the power functions. We now prove this by showing that $x^n \ll e^x$ for every exponent n (Figure 4).

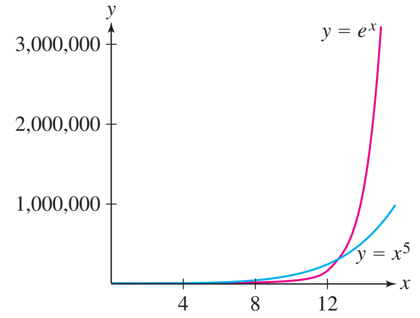


FIGURE 4 Graph illustrating that $x^5 \ll e^x$.

THEOREM 3 Growth of $f(x) = e^x$

$$x^n \ll e^x \qquad \text{for every exponent } n$$

In other words, $\lim_{x\to\infty} \dfrac{e^x}{x^n} = \infty$ for all n.

Proof We first prove the theorem for positive integers n.

$$\lim_{x\to\infty} \frac{e^x}{x^n} = \lim_{x\to\infty} \frac{e^x}{nx^{n-1}} = \lim_{x\to\infty} \frac{e^x}{n(n-1)x^{n-2}} = \dots = \lim_{x\to\infty} \frac{e^x}{n}$$

We applied L'Hôpital's Rule n times, each time obtaining an indeterminate form $\frac{\infty}{\infty}$, until the last stage shown. In $\lim_{x\to\infty} \frac{e^x}{n}$ the numerator goes to ∞ and the denominator is constant (relative to x). Therefore, that limit is infinite, implying that $\lim_{x\to\infty} \frac{e^x}{x^n} = \infty$ if n is a positive integer.

If n is any exponent, we can choose a natural number k such that $k > n$. It is easy to see that $x^n \ll x^k$, and because we also have $x^k \ll e^n$, it follows that $x^n \ll e^x$ for all exponents n. ■

Proof of L'Hôpital's Rule

A full proof of L'Hôpital's Rule, without simplifying assumptions, is presented in a supplement on the text's Web site.

We prove L'Hôpital's Rule here only in the first case of Theorem 1—namely, in the case that $f(a) = g(a) = 0$. We also assume that f' and g' are continuous at $x = a$ and that $g'(a) \neq 0$. Then $g(x) \neq g(a)$ for x near a, but not equal to a, and

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{\dfrac{f(x) - f(a)}{x - a}}{\dfrac{g(x) - g(a)}{x - a}}$$

By the Quotient Law for Limits and the definition of the derivative,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x\to a} \dfrac{f(x) - f(a)}{x - a}}{\lim\limits_{x\to a} \dfrac{g(x) - g(a)}{x - a}} = \frac{f'(a)}{g'(a)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} \qquad ■$$

4.5 SUMMARY

- L'Hôpital's Rule: Assume that f and g are differentiable near a and that

$$f(a) = g(a) = 0$$

  Assume also that $g'(x) \neq 0$ (except possibly at a). Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

  provided that the limit on the right exists or is infinite (∞ or $-\infty$).
- L'Hôpital's Rule applies to indeterminate forms $0/0$ and $\pm\infty/\infty$. It can also apply to limits in any of the forms $0 \cdot \infty$, $\infty - \infty$, 0^0, 1^∞, and ∞^0 by converting the expression to one in either the form $0/0$ or the form $\pm\infty/\infty$.
- L'Hôpital's Rule also applies to limits as $x \to \infty$ or $x \to -\infty$.
- In comparing the growth rates of functions, we say that $f(x)$ grows faster than $g(x)$, and we write $g \ll f$, if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$$

4.5 EXERCISES

Preliminary Questions

1. What is wrong with applying L'Hôpital's Rule to $\lim_{x\to 0} \frac{x^2-2x}{3x-2}$?

2. Does L'Hôpital's Rule apply to $\lim_{x\to a} f(x)g(x)$ if $f(x)$ and $g(x)$ both approach ∞ as $x \to a$?

3. What is wrong with saying, "To apply L'Hôpital's Rule to the limit $\lim_{x\to 0} \frac{\ln(1-x)}{x}$, use the Quotient Rule to differentiate $\frac{\ln(1-x)}{x}$ and then take the limit."

4. What is wrong with applying L'Hôpital's Rule to $\lim_{x\to 0^+} x^{\frac{1}{x}}$?

5. What property of the function $f(x) = e^x$ allows us to say

$$\lim_{x\to a} e^{g(x)} = e^{\lim_{x\to a} g(x)}?$$

Exercises

In Exercises 1–10, evaluate the limit, using L'Hôpital's Rule where it applies.

1. $\lim_{x\to 3} \frac{2x^2-5x-3}{x-4}$

2. $\lim_{x\to -5} \frac{x^2-25}{5-4x-x^2}$

3. $\lim_{x\to 4} \frac{x^3-64}{x^2+16}$

4. $\lim_{x\to -1} \frac{x^4+2x+1}{x^5-2x-1}$

5. $\lim_{x\to 9} \frac{x^{1/2}+x-6}{x^{3/2}-27}$

6. $\lim_{x\to 3} \frac{\sqrt{x+1}-2}{x^3-7x-6}$

7. $\lim_{x\to 0} \frac{\sin 4x}{x^2+3x+1}$

8. $\lim_{x\to 0} \frac{x^3}{\sin x - x}$

9. $\lim_{x\to 0} \frac{\cos 2x - 1}{\sin 5x}$

10. $\lim_{x\to 0} \frac{\cos x - \sin^2 x}{\sin x}$

In Exercises 11–16, use L'Hôpital's Rule to evaluate the limit.

11. $\lim_{x\to\infty} \frac{9x+4}{3-2x}$

12. $\lim_{x\to -\infty} x \sin \frac{1}{x}$

13. $\lim_{x\to\infty} \frac{\ln x}{x^{1/2}}$

14. $\lim_{x\to\infty} \frac{x}{e^x}$

15. $\lim_{x\to -\infty} \frac{\ln(x^4+1)}{x}$

16. $\lim_{x\to\infty} \frac{x^2}{e^x}$

In Exercises 17–54, evaluate the limit.

17. $\lim_{x\to 1} \frac{\sqrt{8+x}-3x^{1/3}}{x^2-3x+2}$

18. $\lim_{x\to 4} \left[\frac{1}{\sqrt{x}-2} - \frac{4}{x-4}\right]$

19. $\lim_{x\to -\infty} \frac{3x-2}{1-5x}$

20. $\lim_{x\to\infty} \frac{x^{2/3}+3x}{x^{5/3}-x}$

21. $\lim_{x\to -\infty} \frac{7x^2+4x}{9-3x^2}$

22. $\lim_{x\to\infty} \frac{3x^3+4x^2}{4x^3-7}$

23. $\lim_{x\to 1} \frac{(1+3x)^{1/2}-2}{(1+7x)^{1/3}-2}$

24. $\lim_{x\to 8} \frac{x^{5/3}-2x-16}{x^{1/3}-2}$

25. $\lim_{x\to 0} \frac{\sin 2x}{\sin 7x}$

26. $\lim_{x\to 0} \frac{\tan 4x}{\tan 5x}$

27. $\lim_{x\to 0} \frac{\tan x}{x}$

28. $\lim_{x\to 0} \left(\cot x - \frac{1}{x}\right)$

29. $\lim_{x\to 0} \frac{\sin x - x\cos x}{x - \sin x}$

30. $\lim_{x\to \pi/2} \left(x - \frac{\pi}{2}\right) \tan x$

31. $\lim_{x\to 0} \frac{\cos(x+\frac{\pi}{2})}{\sin x}$

32. $\lim_{x\to 0} \frac{x^2}{1-\cos x}$

33. $\lim_{x\to \pi/2} \frac{\cos x}{\sin(2x)}$

34. $\lim_{x\to 0} \left(\frac{1}{x^2} - \csc^2 x\right)$

35. $\lim_{x\to \pi/2} (\sec x - \tan x)$

36. $\lim_{x\to 2} \frac{e^{x^2}-e^4}{x-2}$

37. $\lim_{x\to 1} \tan\left(\frac{\pi x}{2}\right) \ln x$

38. $\lim_{x\to 1} \frac{x(\ln x - 1)+1}{(x-1)\ln x}$

39. $\lim_{x\to 0} \frac{e^x-1}{\sin x}$

40. $\lim_{x\to 1} \frac{e^x-e}{\ln x}$

41. $\lim_{x\to 0} \frac{e^{2x}-1-x}{x^2}$

42. $\lim_{x\to\infty} \frac{e^{2x}-1-x}{x^2}$

43. $\lim_{t\to 0^+} (\sin t)(\ln t)$

44. $\lim_{x\to\infty} e^{-x}(x^3-x^2+9)$

45. $\lim_{x\to 0} \frac{a^x-1}{x} \quad (a > 0)$

46. $\lim_{x\to\infty} x^{1/x^2}$

47. $\lim_{x\to 1} (1+\ln x)^{1/(x-1)}$

48. $\lim_{x\to 0^+} x^{\sin x}$

49. $\lim_{x\to 0} (\cos x)^{3/x^2}$

50. $\lim_{x\to\infty} \left(\frac{x}{x+1}\right)^x$

51. $\lim_{x\to 0} \frac{\sin^{-1} x}{x}$

52. $\lim_{x\to 0} \frac{\tan^{-1} x}{\sin^{-1} x}$

53. $\lim_{x\to 1} \frac{\tan^{-1} x - \frac{\pi}{4}}{\tan \frac{\pi}{4}x - 1}$

54. $\lim_{x\to 0^+} \ln x \tan^{-1} x$

55. Evaluate $\lim_{x\to \pi/2} \frac{\cos mx}{\cos nx}$, where $m, n \neq 0$ are integers.

56. Evaluate $\lim_{x\to 1} \frac{x^m-1}{x^n-1}$ for any numbers $m, n \neq 0$.

57. Evaluate each of the following limits.

(a) $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^{x^2}$

(b) $\lim_{x\to\infty} \left(1+\frac{1}{x^2}\right)^x$

58. Show that $\lim_{x\to\infty} \left(1+\frac{r}{x}\right)^x = e^r$.

In Exercises 59–60, a ball is launched straight up in the air and is acted on by air resistance and gravity as in Example 5. The function M gives the maximum height that the projectile attains as a function of the air resistance parameter k. In each case, determine the maximum height as we let the air resistance term go to zero; that is, determine $\lim_{k\to 0} M(k)$.

59. A ball with a mass of 1 kilogram is launched upward with an initial velocity of 60 m/s, and

$$M(k) = \frac{60k - 9.8\ln(\frac{300k}{49} + 1)}{k^2}$$

(Compare with Exercises 37 in Section 2.5 and 29 in Section 3.4.)

60. A ball with a mass of 500 grams is launched upward with an initial velocity of 30 m/s, and

$$M(k) = \frac{15k - 2.45\ln(\frac{300k}{49} + 1)}{k^2}$$

(Compare with Exercise 38 in Section 2.5.)

61. In each case, show that the form is indeterminate by showing that if $\lim_{x\to c} f(x)^{g(x)}$ has the form, then the limit in the exponent in $e^{\lim_{x\to c} g(x)\ln f(x)}$ has a known indeterminate form.

(a) 1^∞

(b) ∞^0

62. GU Can L'Hôpital's Rule be applied to $\lim_{x\to 0^+} x^{\sin(1/x)}$? Does a graphical or numerical investigation suggest that the limit exists?

63. Let $f(x) = x^{1/x}$ for $x > 0$.

(a) Calculate $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to\infty} f(x)$.

(b) Find the maximum value of f and determine the intervals on which f is increasing or decreasing.

64. (a) Use the results of Exercise 63 to prove that $x^{1/x} = c$ has a unique solution if $0 < c \le 1$ or $c = e^{1/e}$, and has two solutions if $1 < c < e^{1/e}$, and no solutions if $c > e^{1/e}$.

(b) GU Plot the graph of $f(x) = x^{1/x}$ and verify that it confirms the conclusions of (a).

65. Determine whether $f \ll g$ or $g \ll f$ (or neither) for the functions $f(x) = \log_{10} x$ and $g(x) = \ln x$.

66. Show that $(\ln x)^3 \ll x^{1/3}$ and $(\ln x)^4 \ll x^{1/10}$.

67. Just as exponential functions are distinguished by their rapid rate of increase, the logarithm functions grow particularly slowly. Show that $\ln x \ll x^a$ for all $a > 0$.

68. Show that $(\ln x)^N \ll x^a$ for all N and all $a > 0$.

69. Determine whether $\sqrt{x} \ll e^{\sqrt{\ln x}}$ or $e^{\sqrt{\ln x}} \ll \sqrt{x}$. *Hint:* Use the substitution $u = \ln x$ instead of L'Hôpital's Rule.

70. Show that $\lim_{x\to\infty} x^n e^{-x} = 0$ for all whole numbers $n > 0$.

71. Assumptions Matter Suppose $f(x) = x(2 + \sin x)$ and let $g(x) = x^2 + 1$.

(a) Show directly that $\lim_{x\to\infty} f(x)/g(x) = 0$.

(b) Show that $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$, but $\lim_{x\to\infty} f'(x)/g'(x)$ does not exist.

Do (a) and (b) contradict L'Hôpital's Rule? Explain.

72. Let $H(b) = \lim_{x\to\infty} \dfrac{\ln(1 + b^x)}{x}$ for $b > 0$.

(a) Show that $H(b) = \ln b$ if $b \ge 1$.

(b) Determine $H(b)$ for $0 < b \le 1$.

73. Let $G(b) = \lim_{x\to\infty} (1 + b^x)^{1/x}$.

(a) Use the result of Exercise 72 to evaluate $G(b)$ for all $b > 0$.

(b) GU Verify your result graphically by plotting $y = (1 + b^x)^{1/x}$ together with the horizontal line $y = G(b)$ for the values $b = 0.25, 0.5, 2, 3$.

74. Show that $\lim_{t\to\infty} t^k e^{-t^2} = 0$ for all k. *Hint:* Compare with $\lim_{t\to\infty} t^k e^{-t} = 0$.

In Exercises 75–77, let

$$f(x) = \begin{cases} e^{-1/x^2} & \textit{for } x \ne 0 \\ 0 & \textit{for } x = 0 \end{cases}$$

These exercises show that f has an unusual property: All of its derivatives at x = 0 exist and are equal to zero.

75. Show that $\lim_{x\to 0} \dfrac{f(x)}{x^k} = 0$ for all k. *Hint:* Let $t = x^{-1}$ and apply the result of Exercise 74.

76. Show that $f'(0)$ exists and is equal to zero. Also, verify that $f''(0)$ exists and is equal to zero.

77. Show that for $k \ge 1$ and $x \ne 0$,

$$f^{(k)}(x) = \frac{P(x)e^{-1/x^2}}{x^r}$$

for some polynomial $P(x)$ and some exponent $r \ge 1$. Use the result of Exercise 75 to show that $f^{(k)}(0)$ exists and is equal to zero for all $k \ge 1$.

Further Insights and Challenges

78. Show that L'Hôpital's Rule applies to $\lim_{x\to\infty} \dfrac{x}{\sqrt{x^2+1}}$ but that it does not help. Then evaluate the limit directly.

79. The Second Derivative Test for critical points fails if $f''(c) = 0$. This exercise develops a **Higher Derivative Test** based on the sign of the first nonzero derivative. Suppose that

$$f'(c) = f''(c) = \cdots = f^{(n-1)}(c) = 0, \quad \text{but} \quad f^{(n)}(c) \ne 0$$

(a) Show, by applying L'Hôpital's Rule n times, that

$$\lim_{x\to c} \frac{f(x) - f(c)}{(x-c)^n} = \frac{1}{n!} f^{(n)}(c)$$

where $n! = n(n-1)(n-2)\cdots(2)(1)$.

(b) Use (a) to show that if n is even, then $f(c)$ is a local minimum if $f^{(n)}(c) > 0$ and is a local maximum if $f^{(n)}(c) < 0$. *Hint:* If n is even, then $(x-c)^n > 0$ for $x \ne a$, so $f(x) - f(c)$ must be positive for x near c if $f^{(n)}(c) > 0$.

(c) Use (a) to show that if n is odd, then $f(c)$ is neither a local minimum nor a local maximum.

80. When a spring with natural frequency $\lambda/2\pi$ is driven with a sinusoidal force $\sin(\omega t)$ with $\omega \ne \lambda$, it oscillates according to

$$y(t) = \frac{1}{\lambda^2 - \omega^2}\big(\lambda\sin(\omega t) - \omega\sin(\lambda t)\big)$$

Let $y_0(t) = \lim_{\omega\to\lambda} y(t)$.

(a) Use L'Hôpital's Rule to determine $y_0(t)$.

(b) Show that $y_0(t)$ ceases to be periodic and that its amplitude $|y_0(t)|$ tends to ∞ as $t \to \infty$ (the system is said to be in **resonance**; eventually, the spring is stretched beyond its structural tolerance).

(c) CAS Plot y for $\lambda = 1$ and $\omega = 0.8$, 0.9, 0.99, and 0.999. Do the graphs confirm your conclusion in (b)?

81. We expended a lot of effort to evaluate $\lim\limits_{x\to 0} \dfrac{\sin x}{x}$ in Chapter 2. Show that we could have evaluated it easily using L'Hôpital's Rule. Then explain why this method would involve *circular reasoning*.

82. By a fact from algebra, if f, g are polynomials such that $f(a) = g(a) = 0$, then there are polynomials f_1, g_1 such that

$$f(x) = (x-a)f_1(x), \qquad g(x) = (x-a)g_1(x)$$

Use this to verify L'Hôpital's Rule directly for $\lim\limits_{x\to a} f(x)/g(x)$.

83. Patience Required Use L'Hôpital's Rule to evaluate and check your answers numerically:

(a) $\lim\limits_{x\to 0^+} \left(\dfrac{\sin x}{x}\right)^{1/x^2}$ **(b)** $\lim\limits_{x\to 0} \left(\dfrac{1}{\sin^2 x} - \dfrac{1}{x^2}\right)$

84. In the following cases, check that $x = c$ is a critical point and use Exercise 79 to determine whether $f(c)$ is a local minimum or a local maximum.

(a) $f(x) = x^5 - 6x^4 + 14x^3 - 16x^2 + 9x + 12 \quad (c = 1)$

(b) $f(x) = x^6 - x^3 \quad (c = 0)$

4.6 Analyzing and Sketching Graphs of Functions

In this section, our goal is to study graphs of functions f using the information provided by the first two derivatives f' and f''. You will see that you can acquire a good understanding of the properties of a graph without plotting a large number of points. Evan though almost all graphs you may see are produced by computer (including, of course, the graphs in this textbook), the tools of calculus provide information beyond the image displayed on a computer. This information includes the exact locations of critical points and inflection points, the rates of increase and decrease over the function's domain, and the concavity of the function.

Most graphs are made up of smaller *arcs* that have one of the four basic shapes, corresponding to the four possible sign combinations of f' and f'' (Figure 1). Since f' and f'' can each have sign $+$ or $-$, the sign combinations are

$$++ \qquad +- \qquad -+ \qquad --$$

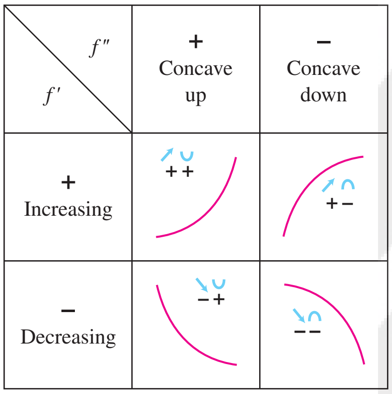


FIGURE 1 The four basic shapes.

In this notation, the first sign refers to f' and the second sign to f''. For instance, $-+$ indicates that $f'(x) < 0$ and $f''(x) > 0$. We use a slanted arrow over the first sign to indicate whether the function is increasing or decreasing, and an upturned or downturned ∪ over the second sign to indicate the concavity.

In analyzing a graph, we focus on the **transition points**, where the basic shape changes due to a sign change in either f' (local min or max) or f'' (point of inflection). In this section, local extrema are indicated by solid dots, and points of inflection are indicated by green solid squares (Figure 2).

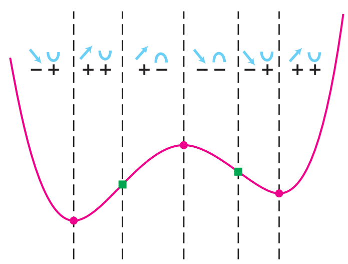


FIGURE 2 The graph of f with transition points and sign combinations of f' and f''.

In examining the properties of a function, it is often useful to investigate the **asymptotic behavior**—that is, the behavior of $f(x)$ as x approaches either $\pm\infty$ or a vertical asymptote.

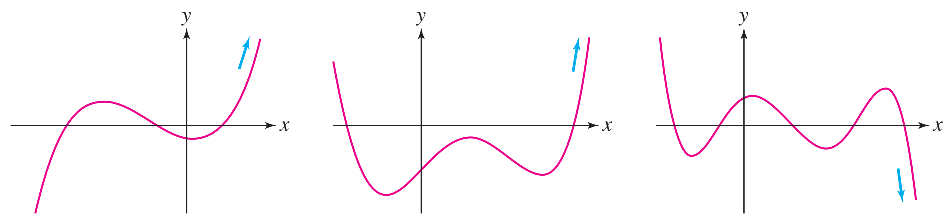
In the examples that follow, we use calculus to investigate the behavior of specific functions, and then we use the information we gather to construct a picture of the function's graph—that is, to "sketch the graph." The first three examples treat polynomials. Recall from Section 2.7 that the limits at infinity of a polynomial

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

(assuming that $a_n \neq 0$) are determined by

$$\lim_{x\to\infty} f(x) = a_n \lim_{x\to\infty} x^n$$

In general, the graph of a polynomial oscillates up and down a finite number of times and tends to positive or negative infinity as x tends to positive or negative infinity. Typical examples appear in Figure 3.



(A) Degree 3, $a_3 > 0$ (B) Degree 4, $a_4 > 0$ (C) Degree 5, $a_5 < 0$

FIGURE 3 Graphs of polynomials.

EXAMPLE 1 **Quadratic Polynomial** Investigate the behavior of $f(x) = x^2 - 4x + 3$ and sketch its graph.

Solution Note that $f(x) = (x-1)(x-3)$ so the graph intersects the x-axis at $x = 1$ and $x = 3$. We have $f'(x) = 2x - 4 = 2(x-2)$. We can see directly that $f'(x)$ is negative for $x < 2$ and positive for $x > 2$, but let's confirm this using test values, as in previous sections:

Interval	Test value	Sign of f'
$(-\infty, 2)$	$f'(1) = -2$	$-$
$(2, \infty)$	$f'(3) = 2$	$+$

Furthermore, $f''(x) = 2$ is positive, so the graph is everywhere concave up. To sketch the graph, plot the local minimum $(2, -1)$, the y-intercept, and the roots $x = 1, 3$. Since the leading term of f is x^2, $f(x)$ tends to ∞ as $x \to \pm\infty$. This asymptotic behavior is noted by the arrows in Figure 4. ■

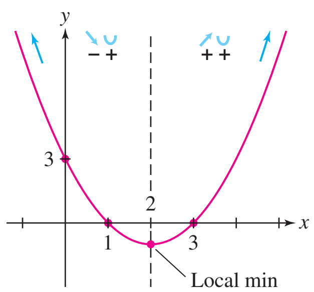


FIGURE 4 Graph of $f(x) = x^2 - 4x + 3$.

EXAMPLE 2 **Cubic Polynomial** Investigate the behavior of the cubic function $f(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x + 3$ and sketch the graph.

Solution

Step 1. **Determine the signs of f' and f''.**
First, solve for the critical points:

$$f'(x) = x^2 - x - 2 = (x+1)(x-2)$$

The critical points are $c = -1, 2$, and they divide the x-axis into three intervals $(-\infty, -1)$, $(-1, 2)$, and $(2, \infty)$, on which we determine the sign of f' by computing test values:

Interval	Test value	Sign of f'
$(-\infty, -1)$	$f'(-2) = 4$	$+$
$(-1, 2)$	$f'(0) = -2$	$-$
$(2, \infty)$	$f'(3) = 4$	$+$

Next, $f''(x) = 2x - 1$, and therefore $x = \frac{1}{2}$ is the only solution to $f''(x) = 0$. We have

Interval	Test value	Sign of f''
$(-\infty, \frac{1}{2})$	$f''(0) = -1$	$-$
$(\frac{1}{2}, \infty)$	$f''(1) = 1$	$+$

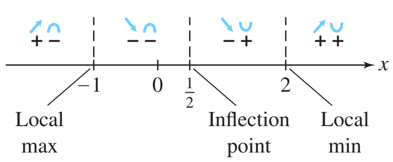


FIGURE 5 Sign combinations of f' and f''.

Step 2. **Note transition points and sign combinations.**
This step merges the information about f' and f'' in a sign diagram (Figure 5). There are three transition points:

- $c = -1$: local max since f' changes from $+$ to $-$.

- $c = \frac{1}{2}$: corresponds to a point of inflection since f'' changes sign.
- $c = 2$: local min since f' changes from $-$ to $+$.

In Figure 6(A), we plot the transition points and, for added accuracy, the y-intercept $f(0)$, using the values

$$f(-1) = \frac{25}{6}, \qquad f\left(\frac{1}{2}\right) = \frac{23}{12}, \qquad f(0) = 3, \qquad f(2) = -\frac{1}{3}$$

Step 3. **Draw arcs of appropriate shape and asymptotic behavior.**
The leading term of $f(x)$ is $\frac{1}{3}x^3$. Therefore, $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = -\infty$.

To create the sketch, it remains only to connect the transition points by arcs of the appropriate concavity and asymptotic behavior, as in Figure 6(B) and (C). ■

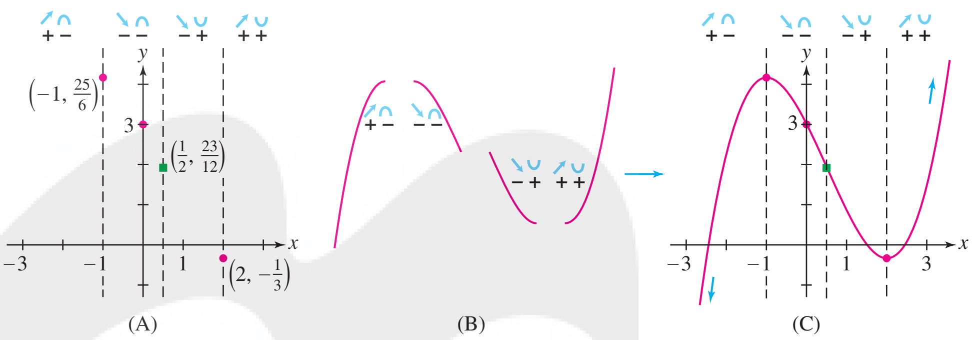


FIGURE 6 Graph of $f(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x + 3$.

EXAMPLE 3 Investigate the behavior of $f(x) = 3x^4 - 8x^3 + 6x^2 + 1$ and sketch its graph.

Solution

Step 1. **Determine the signs of f' and f''.**
First, solve for the transition points:

$$f'(x) = 12x^3 - 24x^2 + 12x = 12x(x-1)^2, \quad \text{so } f' = 0 \quad \Rightarrow \quad x = 0, 1$$

$$f''(x) = 36x^2 - 48x + 12 = 12(x-1)(3x-1), \quad \text{so } f'' = 0 \quad \Rightarrow \quad x = \frac{1}{3}, 1$$

The signs of f' and f'' are recorded in the following tables:

Interval	Test value	Sign of f'
$(-\infty, 0)$	$f'(-1) = -48$	$-$
$(0, 1)$	$f'\left(\frac{1}{2}\right) = \frac{3}{2}$	$+$
$(1, \infty)$	$f'(2) = 24$	$+$

Interval	Test value	Sign of f''
$\left(-\infty, \frac{1}{3}\right)$	$f''(0) = 12$	$+$
$\left(\frac{1}{3}, 1\right)$	$f''\left(\frac{1}{2}\right) = -3$	$-$
$(1, \infty)$	$f''(2) = 60$	$+$

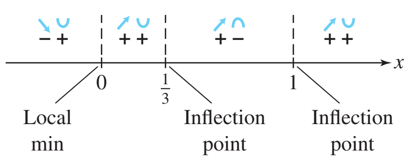


FIGURE 7

Step 2. **Note transition points and sign combinations.**
The transition points $c = 0, \frac{1}{3}, 1$ divide the x-axis into four intervals (Figure 7). The type of sign change determines the nature of the transition point:

- $c = 0$: local min since f' changes from $-$ to $+$.
- $c = \frac{1}{3}$: corresponds to a point of inflection since f'' changes sign.
- $c = 1$: neither a local min nor a local max since f' does not change sign, but it is a point of inflection since $f''(x)$ changes sign.

We plot the transition points $c = 0, \frac{1}{3}, 1$ in Figure 8(A) using function values $f(0) = 1$, $f\left(\frac{1}{3}\right) = \frac{38}{27}$, and $f(1) = 2$.

Step 3. **Draw arcs of appropriate shape and asymptotic behavior.**
Before drawing the arcs, we note that $f(x)$ has leading term $3x^4$, so $f(x)$ tends to ∞ as $x \to \infty$ and as $x \to -\infty$. We obtain Figure 8(B). ■

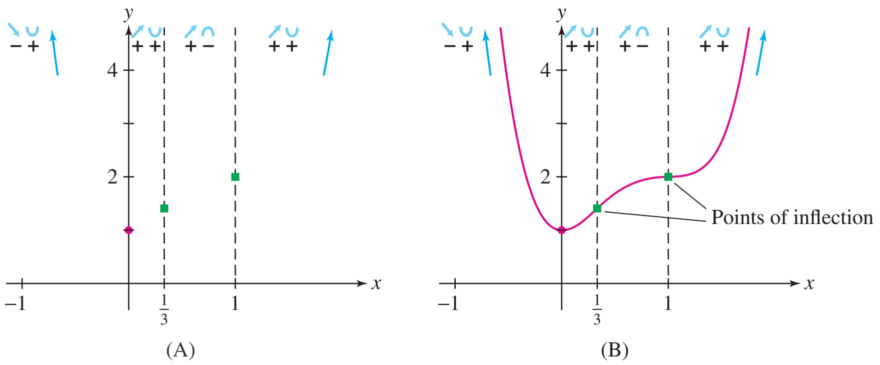


FIGURE 8 $f(x) = 3x^4 - 8x^3 + 6x^2 + 1$.

EXAMPLE 4 Investigate the behavior of $f(x) = \cos x + \frac{1}{2}x$ over $[0, \pi]$, and sketch its graph.

Solution First, we find the transition points for x in $[0, \pi]$:

$$f'(x) = -\sin x + \frac{1}{2}, \quad \text{so } f'(x) = 0 \quad \Rightarrow \quad x = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$f''(x) = -\cos x, \quad \text{so } f''(x) = 0 \quad \Rightarrow \quad x = \frac{\pi}{2}$$

The sign combinations are shown in the following tables:

Interval	Test value	Sign of f'
$(0, \frac{\pi}{6})$	$f'(\frac{\pi}{12}) \approx 0.24$	$+$
$(\frac{\pi}{6}, \frac{5\pi}{6})$	$f'(\frac{\pi}{2}) = -\frac{1}{2}$	$-$
$(\frac{5\pi}{6}, \pi)$	$f'(\frac{11\pi}{12}) \approx 0.24$	$+$

Interval	Test value	Sign of f''
$(0, \frac{\pi}{2})$	$f''(\frac{\pi}{4}) = -\frac{\sqrt{2}}{2}$	$-$
$(\frac{\pi}{2}, \pi)$	$f''(\frac{3\pi}{4}) = \frac{\sqrt{2}}{2}$	$+$

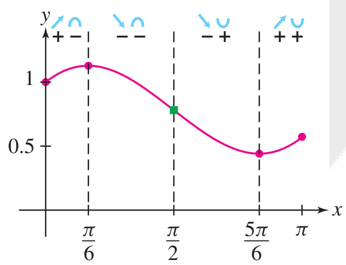


FIGURE 9 $f(x) = \cos x + \frac{1}{2}x$.

We record the sign changes and transition points in Figure 9 and sketch the graph using the values

$$f(0) = 1, \quad f\left(\frac{\pi}{6}\right) \approx 1.13, \quad f\left(\frac{\pi}{2}\right) \approx 0.79, \quad f\left(\frac{5\pi}{6}\right) \approx 0.44, \quad f(\pi) \approx 0.57$$ ■

EXAMPLE 5 Investigate the behavior of $f(x) = xe^x$ and sketch its graph.

Solution As usual, we solve for the transition points and determine the signs:

$$f'(x) = xe^x + e^x = (x+1)e^x, \quad \text{so } f'(x) = 0 \quad \Rightarrow \quad x = -1$$

$$f''(x) = (x+1)e^x + e^x = (x+2)e^x, \quad \text{so } f''(x) = 0 \quad \Rightarrow \quad x = -2$$

Interval	Test value	Sign of f'
$(-\infty, -1)$	$f'(-2) = -e^{-2}$	$-$
$(-1, \infty)$	$f'(0) = e^0$	$+$

Interval	Test value	Sign of f''
$(-\infty, -2)$	$f''(-3) = -e^{-3}$	$-$
$(-2, \infty)$	$f''(0) = 2e^0$	$+$

The sign change of f' shows that $f(-1)$ is a local min. The sign change of f'' shows that f has a point of inflection at $x = -2$, where the graph changes from concave down to concave up.

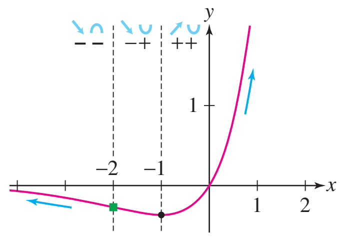


FIGURE 10 Graph of $f(x) = xe^x$. The sign combinations $--$, $-+$, $++$ indicate the signs of f' and f''.

The last pieces of information we need are the limits at infinity. Both x and e^x tend to ∞ as $x \to \infty$, so $\lim\limits_{x\to\infty} xe^x = \infty$. On the other hand, the limit as $x \to -\infty$ is indeterminate of type $\infty \cdot 0$ because x tends to $-\infty$ and e^x tends to zero. Therefore, we write $xe^x = x/e^{-x}$ and apply L'Hôpital's Rule:

$$\lim_{x\to-\infty} xe^x = \lim_{x\to-\infty} \frac{x}{e^{-x}} = \lim_{x\to-\infty} \frac{1}{-e^{-x}} = -\lim_{x\to-\infty} e^x = 0$$

Figure 10 shows the graph with its local minimum and point of inflection, drawn with the correct concavity and asymptotic behavior. ■

EXAMPLE 6 Investigate the behavior of $f(x) = \dfrac{3x+2}{2x-4}$ and sketch its graph.

Solution The function f is not defined for all x. This plays a role in our analysis so we add a Step 0 to our procedure.

***Step 0.* Determine the domain of f.**
Since $f(x)$ is not defined for $x = 2$, the domain of f consists of the two intervals $(-\infty, 2)$ and $(2, \infty)$. We must analyze f on these intervals separately.

***Step 1.* Determine the signs of f' and f''.**
Calculation shows that

$$f'(x) = -\frac{4}{(x-2)^2}, \qquad f''(x) = \frac{8}{(x-2)^3}$$

Although $f'(x)$ is not defined at $x = 2$, it is not a critical point because $x = 2$ is not in the domain of f. In fact, $f'(x)$ is negative for $x \neq 2$, so f is decreasing and has no critical points.

On the other hand, $f''(x) > 0$ for $x > 2$ and $f''(x) < 0$ for $x < 2$, so the concavity of f changes at $x = 2$. However, there is not an inflection point at $x = 2$ because—as was the case above—$x = 2$ is not in the domain of f.

***Step 2.* Note transition points and sign combinations.**
There are no transition points in the domain of f.

$(-\infty, 2)$	$f'(x) < 0$ and $f''(x) < 0$
$(2, \infty)$	$f'(x) < 0$ and $f''(x) > 0$

***Step 3.* Draw arcs of appropriate shape and asymptotic behavior.**
The following limits as $x \to \pm\infty$, evaluated using L'Hôpital's Rule, show that $y = \frac{3}{2}$ is a horizontal asymptote:

$$\lim_{x\to\pm\infty} \frac{3x+2}{2x-4} = \lim_{x\to\pm\infty} \frac{3}{2} = \frac{3}{2}$$

The line $x = 2$ is a vertical asymptote because $f(x)$ has infinite one-sided limits

$$\lim_{x\to 2^-} \frac{3x+2}{2x-4} = -\infty, \qquad \lim_{x\to 2^+} \frac{3x+2}{2x-4} = \infty$$

To verify this, note that for x near 2, the numerator $3x + 2$ is positive while the denominator $2x - 4$ is small and negative for $x < 2$ and is small and positive for $x > 2$. Figure 11(A) summarizes the asymptotic behavior.

Now, to the left of $x = 2$, the graph is decreasing $[f'(x) < 0]$, is concave down $[f''(x) < 0]$, and approaches the asymptotes. The x-intercept is $x = -\frac{2}{3}$ because $f\left(-\frac{2}{3}\right) = 0$, and the y-intercept is $y = f(0) = -\frac{1}{2}$. We obtain the left part of the graph as shown in Figure 11(B). To the right of $x = 2$, the graph is decreasing $[f'(x) < 0]$, is concave up $[f''(x) > 0]$, and approaches the asymptotes as shown. ■

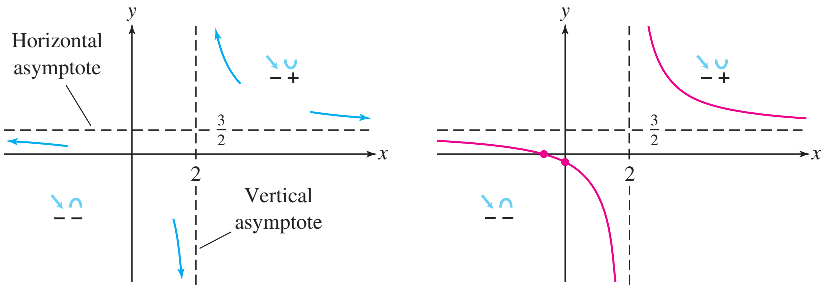


FIGURE 11 Graph of $y = \dfrac{3x+2}{2x-4}$.

EXAMPLE 7 A Logistic Function Analyze the behavior of $P(x) = \frac{50}{1+2e^{-0.1x}}$ and sketch the graph.

Solution The function P is defined for all x. With some careful calculation and simplification, we find that

$$P'(x) = \frac{10e^{-0.1x}}{(1+2e^{-0.1x})^2}, \qquad P''(x) = \frac{e^{-0.1x}(2e^{-0.1x}-1)}{(1+2e^{-0.1x})^3}$$

First, note that $P'(x)$ is defined and positive for all x; therefore P is increasing for all x.

The sign of $P''(x)$ is equal to the sign of $2e^{-0.1x} - 1$ because the denominator and the other factor in the numerator are positive for all x. It follows that $P''(x) = 0$ when $2e^{-0.1x} - 1 = 0$. Solving for x:

$$\begin{aligned} 2e^{-0.1x} &= 1 \\ e^{-0.1x} &= \frac{1}{2} \\ -0.1x &= \ln\frac{1}{2} \\ x &= -10\ln\frac{1}{2} = 10\ln 2 \end{aligned}$$

Thus, $P''(x) = 0$ at $x = 10\ln 2 \approx 6.93$. Furthermore, $P''(x)$ is positive to the left of $10\ln 2$ and is negative to the right. Therefore there is an inflection point at $x = 10\ln 2$.

Figure 12 summarizes the sign information.

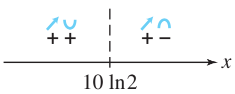


FIGURE 12 Signs of P' and P''.

The lines $P = 0$ and $P = 50$ are horizontal asymptotes because

$$\lim_{x\to\infty} \frac{50}{1+2e^{-0.1x}} = 50 \qquad \text{and} \qquad \lim_{x\to-\infty} \frac{50}{1+2e^{-0.1x}} = 0$$

The graph of P increases away from the asymptote $P = 0$ and is concave up until reaching $x = 10\ln 2 \approx 6.93$. From that point on, it continues to increase but is concave down and approaches the asymptote $P = 50$. Note that $P(10\ln 2) = 25$, so that at the inflection point, we are at half of the limiting value 50. The graph is sketched in Figure 13. ■

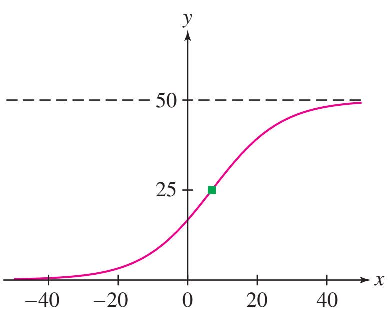


FIGURE 13 The graph of the logistic function $P(x)$.

Properties that we observed for P in the previous example hold for general logistic functions $P(x) = \frac{M}{1+Ae^{-kx}}$ for M, A, and k all positive. In particular (see Exercise 73):

- $\lim\limits_{x\to-\infty} P(x) = 0$ and $\lim\limits_{x\to\infty} P(x) = M$, so P has horizontal asymptotes at $P = 0$ and $P = M$.
- P is increasing for all x.
- There is an inflection point at $\left(\frac{\ln A}{k}, \frac{M}{2}\right)$, and P is concave up to the left of the inflection point, concave down to the right.

If a logistic function is modeling a population, such as in Example 2 in Section 2.7, then these properties show that the population increases at an increasing rate until it equals half of the carrying capacity; beyond that, it continues to increase but at a decreasing rate, approaching the carrying capacity in the long run.

4.6 SUMMARY

- Most graphs are made up of arcs that have one of the four basic shapes (Figure 14):

	Sign combination	Curve type
$++$	$f' > 0,\ f'' > 0$	Increasing and concave up
$+-$	$f' > 0,\ f'' < 0$	Increasing and concave down
$-+$	$f' < 0,\ f'' > 0$	Decreasing and concave up
$--$	$f' < 0,\ f'' < 0$	Decreasing and concave down

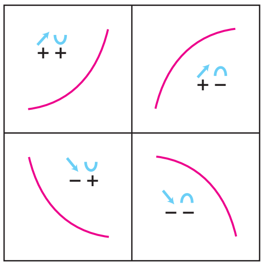


FIGURE 14 The four basic shapes.

- A *transition point* is a point in the domain of f at which either f' changes sign (local min or max) or f'' changes sign (point of inflection).
- It is convenient to break up the curve-sketching process into steps:

Step 0. Determine the domain of f.
Step 1. Determine the signs of f' and f''.
Step 2. Note transition points and sign combinations.
Step 3. Determine the asymptotic behavior of $f(x)$.
Step 4. Draw arcs of appropriate shape and asymptotic behavior.

4.6 EXERCISES

Preliminary Questions

1. Sketch an arc where f' and f'' have the sign combination $++$. Do the same for $-+$.

2. If the sign combination of f' and f'' changes from $++$ to $+-$ at $x = c$, then (choose the correct answer)

(a) $f(c)$ is a local min.

(b) $f(c)$ is a local max.

(c) $(c, f(c))$ is a point of inflection.

3. The second derivative of the function $f(x) = (x-4)^{-1}$ is $f''(x) = 2(x-4)^{-3}$. Although $f''(x)$ changes sign at $x = 4$, f does not have a point of inflection at $x = 4$. Why not?

Exercises

1. Determine the sign combinations of f' and f'' for each interval A–G in Figure 15.

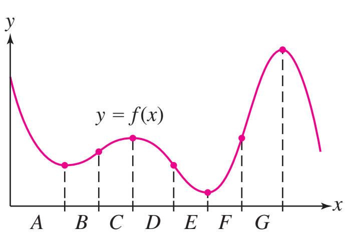


FIGURE 15

2. State the sign change at each transition point A–G in Figure 16. Example: $f'(x)$ goes from $+$ to $-$ at A.

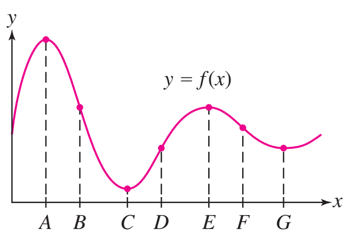


FIGURE 16

In Exercises 3–6, draw the graph of a function for which f' and f'' take on the given sign combinations in order.

3. $++,\ +-,\ --$

4. $+-,\ --,\ -+$

5. $-+,\ --,\ -+$

6. $-+,\ ++,\ +-$

7. Sketch the graph of a function that could have the graphs of f' and f'' appearing in Figure 17.

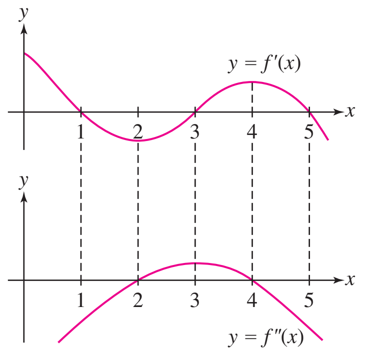


FIGURE 17

8. Sketch the graph of a function that could have the graphs of f' and f'' appearing in Figure 18.

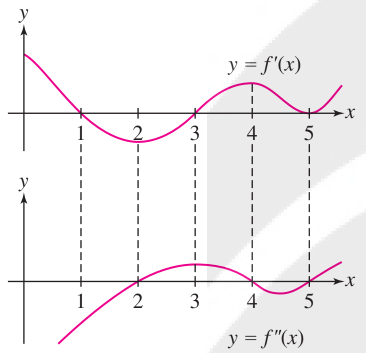


FIGURE 18

9. Investigate the behavior and sketch the graph of $y = x^2 - 5x + 4$.

10. Investigate the behavior and sketch the graph of $y = 12 - 5x - 2x^2$.

11. Investigate the behavior and sketch the graph of $f(x) = x^3 - 3x^2 + 2$. Include the zeros of f, which are $x = 1$ and $1 \pm \sqrt{3}$ (approximately $-0.73, 2.73$).

12. Show that $f(x) = x^3 - 3x^2 + 6x$ has a point of inflection but no local extreme values. Sketch the graph.

13. Extend the sketch of the graph of $f(x) = \cos x + \frac{1}{2}x$ in Example 4 to the interval $[0, 5\pi]$.

14. Investigate the behavior and sketch the graphs of $y = x^{2/3}$ and $y = x^{4/3}$.

In Exercises 15–36, find the transition points, intervals of increase/decrease, concavity, and asymptotic behavior. Then sketch the graph, with this information indicated.

15. $y = x^3 + 24x^2$

16. $y = x^3 - 3x + 5$

17. $y = x^2 - 4x^3$

18. $y = \frac{1}{3}x^3 + x^2 + 3x$

19. $y = 4 - 2x^2 + \frac{1}{6}x^4$

20. $y = 7x^4 - 6x^2 + 1$

21. $y = x^5 + 5x$

22. $y = x^5 - 15x^3$

23. $y = x^4 - 3x^3 + 4x$

24. $y = x^2(x - 4)^2$

25. $y = x^7 - 14x^6$

26. $y = x^6 - 9x^4$

27. $y = x - 4\sqrt{x}$

28. $y = \sqrt{x} + \sqrt{16 - x}$

29. $y = x(8 - x)^{1/3}$

30. $y = (x^2 - 4x)^{1/3}$

31. $y = xe^{-x^2}$

32. $y = (2x^2 - 1)e^{-x^2}$

33. $y = x - 2\ln x$

34. $y = x(4 - x) - 3\ln x$

35. $y = x^2 - 2\ln x$

36. $y = x - 2\ln(x^2 + 1)$

37. Investigate the behavior and sketch the graph of the function $f(x) = 18(x - 3)(x - 1)^{2/3}$ using the formulas

$$f'(x) = \frac{30\left(x - \frac{9}{5}\right)}{(x - 1)^{1/3}}, \qquad f''(x) = \frac{20\left(x - \frac{3}{5}\right)}{(x - 1)^{4/3}}$$

38. Investigate the behavior and sketch the graph of $f(x) = \dfrac{x}{x^2 + 1}$ using the formulas

$$f'(x) = \frac{1 - x^2}{(1 + x^2)^2}, \qquad f''(x) = \frac{2x(x^2 - 3)}{(x^2 + 1)^3}$$

CAS *In Exercises 39–42, sketch the graph of the function, indicating all transition points. If necessary, use a graphing utility or computer algebra system to locate the transition points numerically.*

39. $y = x^2 - 10\ln(x^2 + 1)$

40. $y = e^{-x/2}\ln x$

41. $y = x^4 - 4x^2 + x + 1$

42. $y = 2\sqrt{x} - \sin x, \quad 0 \le x \le 2\pi$

In Exercises 43–48, sketch the graph over the given interval, with all transition points indicated.

43. $y = x + \sin x, \quad [0, 2\pi]$

44. $y = \sin x + \cos x, \quad [0, 2\pi]$

45. $y = 2\sin x - \cos^2 x, \quad [0, 2\pi]$

46. $y = \sin x + \frac{1}{2}x, \quad [0, 2\pi]$

47. $y = \sin x + \sqrt{3}\cos x, \quad [0, \pi]$

48. $y = \sin x - \frac{1}{2}\sin 2x, \quad [0, \pi]$

49. Are all sign transitions possible? Explain with a sketch why the transitions $++ \to -+$ and $-- \to +-$ do not occur if the function is differentiable. (See Exercise 80 for a proof.)

50. Suppose that f is twice differentiable satisfying (i) $f(0) = 1$, (ii) $f'(x) > 0$ for all $x \neq 0$, and (iii) $f''(x) < 0$ for $x < 0$ and $f''(x) > 0$ for $x > 0$. Let $g(x) = f(x^2)$.

(a) Sketch a possible graph of f.

(b) Prove that g has no points of inflection and a unique local extreme value at $x = 0$. Sketch a possible graph of g.

In Exercises 51–52, draw the graph of a function f having the given limits at $\pm\infty$ and for which f' and f'' take on the given sign combinations in order.

51. $\lim_{x\to-\infty} f(x) = -\infty$, $\lim_{x\to\infty} f(x) = 0$; $+-$, $--$, $-+$, $++$, $+-$

52. $\lim_{x\to-\infty} f(x) = -1$, $\lim_{x\to\infty} f(x) = 1$; $++$, $+-$, $--$, $-+$

53. Match the graphs in Figure 19 with the two functions $y = \dfrac{3x}{x^2 - 1}$ and $y = \dfrac{3x^2}{x^2 - 1}$. Explain.

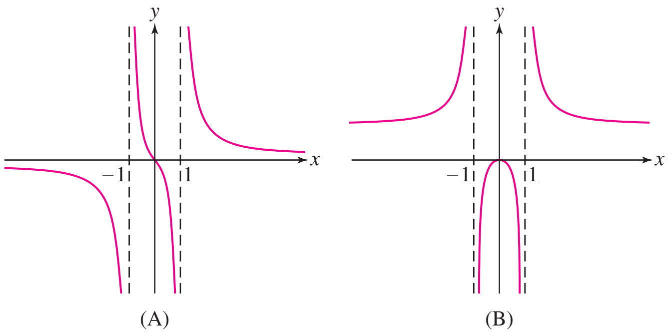


FIGURE 19

54. Match the functions below with their graphs in Figure 20.

(a) $y = \dfrac{1}{x^2 - 1}$ **(b)** $y = \dfrac{x^2}{x^2 + 1}$

(c) $y = \dfrac{1}{x^2 + 1}$ **(d)** $y = \dfrac{x}{x^2 - 1}$

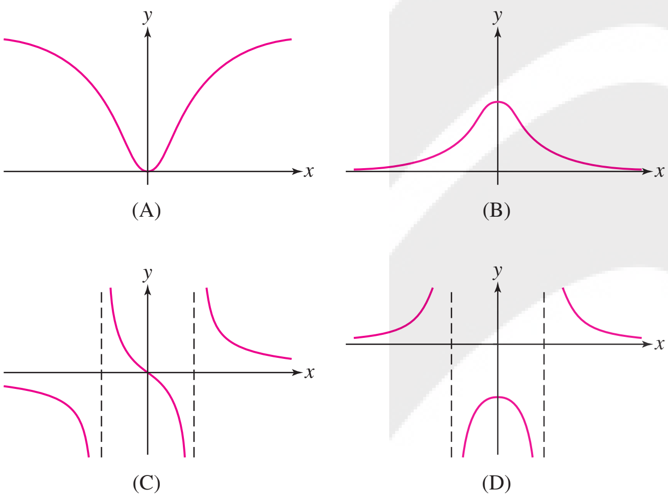


FIGURE 20

In Exercises 55–72, sketch the graph of the function. Indicate the transition points and asymptotes.

55. $y = \dfrac{1}{3x - 1}$

56. $y = \dfrac{x - 2}{x - 3}$

57. $y = \dfrac{x + 3}{x - 2}$

58. $y = x + \dfrac{1}{x}$

59. $y = \dfrac{1}{x} + \dfrac{1}{x - 1}$

60. $y = \dfrac{1}{x} - \dfrac{1}{x - 1}$

61. $y = \dfrac{1}{x(x - 2)}$

62. $y = \dfrac{x}{x^2 - 9}$

63. $y = \dfrac{1}{x^2 - 6x + 8}$

64. $y = \dfrac{x^3 + 1}{x}$

65. $y = 1 - \dfrac{3}{x} + \dfrac{4}{x^3}$

66. $y = \dfrac{1}{x^2} + \dfrac{1}{(x - 2)^2}$

67. $y = \dfrac{1}{x^2} - \dfrac{1}{(x - 2)^2}$

68. $y = \dfrac{4}{x^2 - 9}$

69. $y = \dfrac{1}{(x^2 + 1)^2}$

70. $y = \dfrac{x^2}{(x^2 - 1)(x^2 + 1)}$

71. $y = \dfrac{1}{\sqrt{x^2 + 1}}$

72. $y = \dfrac{x}{\sqrt{x^2 + 1}}$

73. Consider the general logistic function, $P(x) = \frac{M}{1 + Ae^{-kx}}$, with A, M, and k all positive. Show that

(a) $P'(x) = \frac{MAke^{-kx}}{(1+Ae^{-kx})^2}$ and $P''(x) = \frac{MAk^2e^{-kx}(Ae^{-kx}-1)}{(1+Ae^{-kx})^3}$

(b) $\lim\limits_{x\to-\infty} P(x) = 0$ and $\lim\limits_{x\to\infty} P(x) = M$, and therefore $P = 0$ and $P = M$ are horizontal asymptotes of P.

(c) P is increasing for all x.

(d) The only inflection point of P is at $\left(\frac{\ln A}{k}, \frac{M}{2}\right)$. To the left of it P is concave up, and to the right of it P is concave down.

74. Show that the function $R(x) = \frac{M}{\pi}\left(\frac{\pi}{2} + \tan^{-1} x\right)$, with $M > 0$, has the following properties (similar to the general logistic function):

(a) $\lim\limits_{x\to-\infty} R(x) = 0$ and $\lim\limits_{x\to\infty} R(x) = M$, and therefore $R = 0$ and $R = M$ are horizontal asymptotes of R.

(b) R is increasing for all x.

(c) R has a single inflection point. The value of R at the inflection point is $M/2$. To the left of the inflection point R is concave up, to the right R is concave down.

Further Insights and Challenges

*In Exercises 75–79, we explore functions whose graphs approach a nonhorizontal line as $x \to \infty$. A line $y = ax + b$ is called a **slant asymptote** if*

$$\lim_{x\to\infty} (f(x) - (ax + b)) = 0$$

or

$$\lim_{x\to-\infty} (f(x) - (ax + b)) = 0$$

75. Let $f(x) = \dfrac{x^2}{x - 1}$ (Figure 21). Verify the following:

(a) $f(0)$ is a local max and $f(2)$ a local min.

(b) f is concave down on $(-\infty, 1)$ and concave up on $(1, \infty)$.

(c) $\lim\limits_{x\to1-} f(x) = -\infty$ and $\lim\limits_{x\to1+} f(x) = \infty$.

(d) $y = x + 1$ is a slant asymptote of f as $x \to \pm\infty$.

(e) The slant asymptote lies above the graph of f for $x < 1$ and below the graph for $x > 1$.

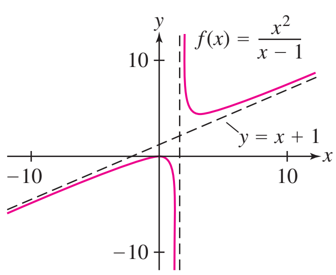


FIGURE 21

76. If $f(x) = P(x)/Q(x)$, where P and Q are polynomials of degrees $m+1$ and m, then by long division, we can write

$$f(x) = (ax+b) + P_1(x)/Q(x)$$

where P_1 is a polynomial of degree $< m$. Show that $y = ax+b$ is the slant asymptote of $f(x)$. Use this procedure to find the slant asymptotes of the following functions:

(a) $y = \dfrac{x^2}{x+2}$ **(b)** $y = \dfrac{x^3+x}{x^2+x+1}$

77. Sketch the graph of

$$f(x) = \frac{x^2}{x+1}$$

Proceed as in the previous exercise to find the slant asymptote.

78. Show that $y = 3x$ is a slant asymptote for $f(x) = 3x + x^{-2}$. Determine whether $f(x)$ approaches the slant asymptote from above or below, and make a sketch of the graph.

79. Sketch the graph of $f(x) = \dfrac{1-x^2}{2-x}$.

80. Assume that f' and f'' exist for all x and let c be a critical point of f. Show that $f(x)$ cannot make a transition from $++$ to $-+$ at $x = c$. *Hint:* Apply the MVT to $f'(x)$.

81. Assume that f'' exists and $f''(x) > 0$ for all x. Show that $f(x)$ cannot be negative for all x. *Hint:* Show that $f'(b) \neq 0$ for some b and use the result of Exercise 74 in Section 4.4.

4.7 Applied Optimization

Optimization plays a role in a wide range of disciplines, including the physical sciences, economics, and biology. For example, scientists have studied how migrating birds choose an optimal velocity v that maximizes the distance D they can travel without stopping, given the energy that can be stored as body fat (Figure 1).

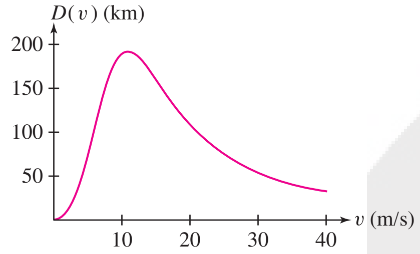


FIGURE 1 Physiology and aerodynamics are applied to obtain a plausible formula for bird migration distance D as a function of velocity v. The optimal velocity corresponds to the maximum point on the graph (see Exercise 69).

In many optimization problems, the first step is to write down the **objective function**. This is the function whose minimum or maximum we seek. Once we find the objective function, we can apply the techniques developed in this chapter. Our first examples require optimization on a closed interval $[a,b]$. Let's recall the steps for finding extrema developed in Section 4.2:

(i) Find the critical points of f in $[a,b]$.
(ii) Evaluate $f(x)$ at the critical points and the endpoints a and b.
(iii) The least and greatest values are the extreme values of f on $[a,b]$.

EXAMPLE 1 A piece of wire of length L is bent into the shape of a rectangle (Figure 2). Which dimensions produce the rectangle of maximum area?

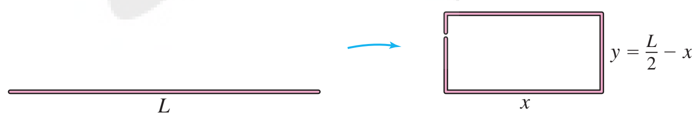


FIGURE 2

Solution The rectangle has area $A = xy$, where x and y are the lengths of the sides. Since A depends on two variables x and y, we cannot find the maximum until we eliminate one of the variables. We can do this because the variables are related: The rectangle has perimeter $L = 2x + 2y$, so $y = \frac{1}{2}L - x$. This allows us to rewrite the area in terms of x alone to obtain the objective function

An equation relating two or more independent variables in an optimization problem is called a constraint equation. The idea is that we cannot assume the variables take on any values we want; instead they are constrained to satisfy a specific equation. In Example 1, the constraint equation is

$$2x + 2y = L$$

$$A(x) = x\left(\frac{1}{2}L - x\right) = \frac{1}{2}Lx - x^2$$

On which interval does the optimization take place? The sides of the rectangle are nonnegative, so we require both $x \geq 0$ and $\frac{1}{2}L - x \geq 0$. Thus, $0 \leq x \leq \frac{1}{2}L$. Our problem is to maximize $A(x)$ on the closed interval $\left[0, \frac{1}{2}L\right]$.

We have $A'(x) = \frac{1}{2}L - 2x$. Solving $A'(x) = 0$, we obtain just a single critical point, $x = \frac{1}{4}L$. Comparing values of A, we find:

$$\text{Endpoints:} \qquad A(0) = 0$$

$$A\left(\frac{1}{2}L\right) = \frac{1}{2}L\left(\frac{1}{2}L - \frac{1}{2}L\right) = 0$$

$$\text{Critical point:} \quad A\left(\frac{1}{4}L\right) = \left(\frac{1}{4}L\right)\left(\frac{1}{2}L - \frac{1}{4}L\right) = \frac{1}{16}L^2$$

The greatest value occurs for $x = \frac{1}{4}L$, and in this case, $y = \frac{1}{2}L - \frac{1}{4}L = \frac{1}{4}L$. The rectangle of maximum area is the square of sides $x = y = \frac{1}{4}L$. ■

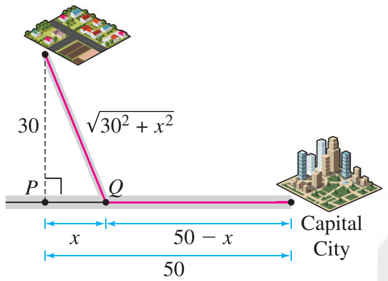


FIGURE 3

EXAMPLE 2 Minimizing Travel Time Your task is to build a road joining the small town of Calverton to Route 1 to enable drivers to reach Capital City in the shortest time (Figure 3). How should this be done if the speed limit is 60 km/hour on the road and 110 km/h on Route 1? The perpendicular distance from Calverton to Route 1 is 30 km, and Capital City is 50 km down Route 1.

Solution We will solve this problem in three steps. These steps can be helpful when solving other optimization problems.

Step 1. **Choose variables.**
We need to determine the point Q where the road will join the Route 1. So let x be the distance from Q to the point P where the perpendicular joins Route 1.

Step 2. **Find the objective function and the interval.**
Our objective function is the time $T(x)$ of the trip as a function of x. To find a formula for $T(x)$, recall that distance traveled at constant velocity v is $d = vt$, and the *time* required to travel a distance d is $t = d/v$. The road has length $\sqrt{30^2 + x^2}$ by the Pythagorean Theorem, so at velocity $v = 60$ km/h, it takes

$$\frac{\sqrt{30^2 + x^2}}{60} \text{ hours to travel from Calverton to } Q$$

The segment of Route 1 from Q to Capital City has length $50 - x$. At velocity $v = 110$ km/h, it takes

$$\frac{50 - x}{110} \text{ h to travel from } Q \text{ to the city}$$

The total number of hours for the trip is

$$T(x) = \frac{\sqrt{30^2 + x^2}}{60} + \frac{50 - x}{110}$$

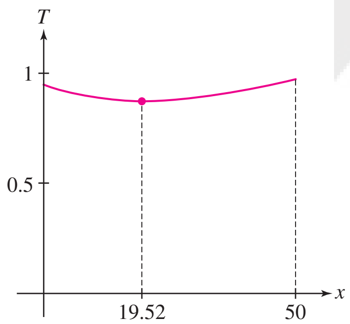


FIGURE 4 Graph of time of trip as function of x.

Our interval is $0 \le x \le 50$ because the road joins Route 1 somewhere between P and Capital City. So our task is to minimize T on $[0, 50]$ (Figure 4).

Step 3. **Optimize.**
Solve for the critical points:

$$T'(x) = \frac{x}{60\sqrt{30^2 + x^2}} - \frac{1}{110} = 0$$

$$110x = 60\sqrt{30^2 + x^2} \quad \Rightarrow \quad 11x = 6\sqrt{30^2 + x^2} \quad \Rightarrow$$

$$121x^2 = 36(30^2 + x^2) \quad \Rightarrow \quad 85x^2 = 32{,}400 \quad \Rightarrow \quad x = \sqrt{32{,}400/85} \approx 19.52$$

To find the minimum value of T, we compare the values of $T(x)$ at the critical point and the endpoints of $[0, 50]$:

$$T(0) \approx 0.95 \text{ h}, \qquad T(19.52) \approx 0.87 \text{ h}, \qquad T(50) \approx 0.97 \text{ h}$$

We conclude that the travel time is minimized if the road joins Route 1 at a distance $x \approx 19.52$ km along the highway from P. ■

EXAMPLE 3 Old Route 1 and Minimizing Travel Time We revisit the situation in Example 2, considering a different pair of speeds along the road and Route 1. Suppose that Route 1 is old and in disrepair and we cannot expect to travel faster than 70 km/hour on it. Furthermore, assume that the new road will be designed for travel at 80 km/h. Now, how should the road be laid out in relation to Route 1?

Solution Intuitively it seems clear that we should have the road go straight from Calverton to Capital City, completely avoiding Route 1. We will work out the solution and see how this case compares with the previous example. Taking the same approach used in the previous example, we find that the task is to determine the minimum of

$$T(x) = \frac{\sqrt{900 + x^2}}{80} + \frac{50 - x}{70}$$

over the interval $[0, 50]$.

In this case, T has no critical points (see Exercise 19). Thus, the minimum of T must occur at one of the endpoints (Figure 5). We have $T(0) \approx 1.09$ h, and $T(50) \approx 0.73$ h. So the minimum of T over $[0, 50]$ occurs at $x = 50$. Therefore, to minimize the time of the trip, the road should go directly from Calverton to Capital City, confirming our initial intuitive analysis. ■

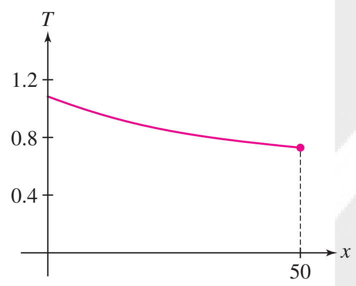


FIGURE 5 In this case $T(x)$ has no critical points in $[0, 50]$.

In Example 2, the minimum occurred at an x between 0 and 50, and in Example 3, it occurred at $x = 50$. It is natural to ask whether there is a combination of speeds along the road and Route 1 so that the minimum occurs at $x = 0$? The answer is no (see Exercise 19). By choosing the Route 1 speed large enough in relation to the road speed, it is possible to have the minimum of T occur at a critical point as close as you like to $x = 0$, but there is no combination of speeds that results in a minimum at exactly $x = 0$.

EXAMPLE 4 Optimal Price All units in a 30-unit apartment building are rented out when the rent is set at $r = \$2000$ per month. A survey reveals that for each \$100 increase in rent, demand for apartments will decrease, such that one additional apartment becomes vacant. Suppose that each occupied unit costs \$200 per month in maintenance. Which rent r maximizes monthly profit?

Solution

Step 1. **Choose variables.**

Our goal is to maximize the total monthly profit P. Let r be the monthly rent and let $N(r)$ be the number of occupied units when the rent is set at r.

Step 2. **Find the objective function and the interval.**

Since one unit becomes vacant with each \$100 increase in rent above \$2000, we find that $(r - 2000)/100$ units are vacant when $r > 2000$. Therefore,

$$N(r) = 30 - \frac{1}{100}(r - 2000) = 50 - \frac{1}{100}r$$

Total monthly profit is equal to the number of occupied units times the profit per unit, which is $r - 200$ (because each unit costs \$200 in maintenance), so

$$P(r) = N(r)(r - 200) = \left(50 - \frac{1}{100}r\right)(r - 200) = -10{,}000 + 52r - \frac{1}{100}r^2$$

Which interval of r-values should we consider? There is no reason to lower the rent below $r = 2000$ because all units are already occupied when $r = 2000$. On the other hand, for the upper limit of r we take the rent at which no units are occupied; that is, the r for which $N(r) = 0$. That occurs at $r = 100 \cdot 50 = 5000$. Therefore, we consider $P(r)$ over the interval $2000 \leq r \leq 5000$.

Step 3. **Optimize.**

Solve for the critical points:

$$P'(r) = 52 - \frac{1}{50}r \quad \text{so } P'(r) = 0 \quad \Rightarrow \quad r = 2600$$

and compare values at the critical point and the endpoints:

$$P(2000) = 54{,}000, \qquad P(2600) = 62{,}400, \qquad P(5000) = 0$$

We conclude that the profit is maximized when the rent is set at $r = \$2600$. In this case, 24 units are occupied. Note that if the maximum profit had occurred at a price that gave us a fractional number of units occupied, we could not have achieved that maximum. Instead, we would have taken the price corresponding to rounding the fractional number up or down to the integer number of units that maximized our profit. ■

Open Versus Closed Intervals

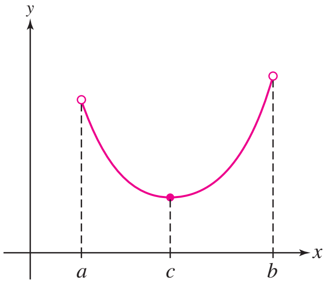


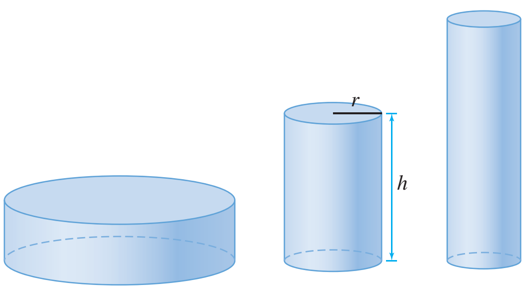
FIGURE 6 A function with a minimum but no maximum over the open interval (a, b).

In contrast to the case of a closed interval, when optimizing a function over an open interval, there is no guarantee that a min or max exists. For example, in Figure 6, a minimum exists at $x = c$ but there is no maximum value. As we approach the endpoint at b, the function values increase, but there is no maximumum because b is not included in the interval (and furthermore the function is not defined there).

If a min or max does exist on an open interval, then it must occur at a critical point (because it is also a local min or max).

With a closed interval, to search for a min and max, we need to evaluate the function at the endpoints of the interval. With an open interval, we need to examine the behavior of the function as x approachs the endpoints of the interval in order to make conclusions about the existence (or lack thereof) of max values and min values. For example, if $f(x)$ tends to infinity at the endpoints, then there is no maximum, and a minimum must occur at a critical point somewhere in the interval. We consider such a situation in the next example.

EXAMPLE 5 Design a cylindrical can of volume 900 cm^3 so that it uses the least amount of metal (Figure 7). In other words, minimize the surface area of the can (including its top and bottom).



DF FIGURE 7 Cylinders with the same volume but different surface areas.

Solution

Step 1. **Choose variables.**

We want to find the radius and the height of the can with minimum surface area. Therefore, we let r be the radius and h the height. Furthermore, we denote the surface area of the can by A.

***Step 2.* Find the objective function and the interval.**

We express A as a function of r and h:

$$A = \underbrace{\pi r^2}_{\text{Top}} + \underbrace{\pi r^2}_{\text{Bottom}} + \underbrace{2\pi r h}_{\text{Side}} = 2\pi r^2 + 2\pi r h$$

The can's volume is $V = \pi r^2 h$. Since we require that $V = 900$ cm^3, we have the constraint equation $\pi r^2 h = 900$. Thus, $h = (900/\pi) r^{-2}$ and

$$A(r) = 2\pi r^2 + 2\pi r\left(\frac{900}{\pi r^2}\right) = 2\pi r^2 + \frac{1800}{r}$$

The radius r can take on any positive value, so we minimize $A(r)$ on $(0, \infty)$.

***Step 3.* Optimize the function.**

Observe that $A(r)$ tends to infinity as r approaches the endpoints of $(0, \infty)$:

- $A(r) \to \infty$ as $r \to \infty$ (because of the r^2 term).
- $A(r) \to \infty$ as $r \to 0$ (because of the $1/r$ term).

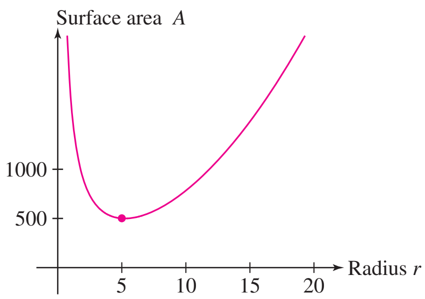


FIGURE 8 Surface area increases as r tends to 0 or ∞. The minimum value exists.

Therefore, $A(r)$ must take on a minimum value at a critical point in $(0, \infty)$ (Figure 8). We solve in the usual way:

$$\frac{dA}{dr} = 4\pi r - \frac{1800}{r^2} = 0 \quad \Rightarrow \quad r^3 = \frac{450}{\pi} \quad \Rightarrow \quad r = \left(\frac{450}{\pi}\right)^{1/3} \approx 5.23 \text{ cm}$$

In the case of a single critical point, as we have here, a second method for proving that the point corresponds to a minimum is to apply the First Derivative Test. Since $A'(r) < 0$ for $r < \left(\frac{450}{\pi}\right)^{1/3}$ and $A'(r) > 0$ for $r > \left(\frac{450}{\pi}\right)^{1/3}$, the critical point must be a local minimum and as the only critical point, the global minimum. A third method would be to apply the Second Derivative Test to show this is a local minimum and therefore, as the only extreme point, the global minimum.

We also need to calculate the height:

$$h = \frac{900}{\pi r^2} = 2\left(\frac{450}{\pi}\right) r^{-2} = 2\left(\frac{450}{\pi}\right)\left(\frac{450}{\pi}\right)^{-2/3} = 2\left(\frac{450}{\pi}\right)^{1/3} \approx 10.46 \text{ cm}$$

Since we have a single critical point in our interval, it follows that we obtain the minimum of A there. Thus, the minimum surface area occurs when a can has radius approximately 5.23 cm and height approximately 10.46 cm. Notice that the optimal dimensions satisfy $h = 2r$. In other words, the optimal can is as tall as it is wide. ■

EXAMPLE 6 **Optimization Problem with No Solution** Is it possible to design a cylinder of volume 900 cm^3 with the largest possible surface area?

Solution The answer is no. In the previous example, we showed that a cylinder of volume 900 cm^3 and radius r has surface area

$$A(r) = 2\pi r^2 + \frac{1800}{r}$$

This function has no maximum value because it tends to infinity as $r \to 0$ or $r \to \infty$ (Figure 8). This means that a cylinder of fixed volume has a large surface area if it is either very fat and short (r large) or very tall and skinny (r small). ■

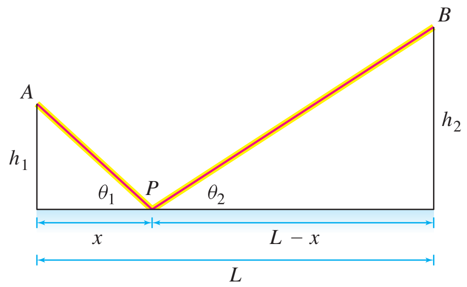
*The Principle of Least Distance is also called **Heron's Principle** after the mathematician Heron of Alexandria (c. 100 CE). See Exercise 81 for an elementary proof that does not use calculus and would have been known to Heron. Exercise 56 develops Snell's Law, a more general optical law based on the Principle of Least Time.*

The **Principle of Least Distance** states that a light beam reflected in a mirror travels along the shortest path. More precisely, a beam traveling from A to B, as in Figure 9, is reflected at the point P for which the path APB has minimum length. In the next example, we show that this minimum occurs when *the angle of incidence is equal to the angle of reflection*, that is, $\theta_1 = \theta_2$.

EXAMPLE 7 Show that if P is the point for which the path APB in Figure 9 has minimal length, then $\theta_1 = \theta_2$.

Solution By the Pythagorean Theorem, the path APB has length

$$f(x) = AP + PB = \sqrt{x^2 + h_1^2} + \sqrt{(L - x)^2 + h_2^2}$$



DF **FIGURE 9** Reflection of a light beam in a mirror.

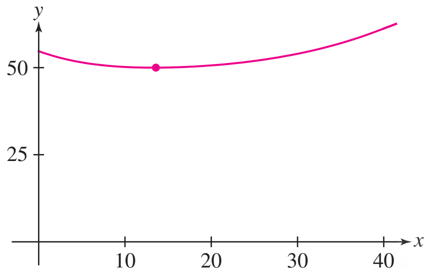


FIGURE 10 Graph of path length for $h_1 = 10$, $h_2 = 20$, $L = 40$.

with x, h_1, and h_2 as in the figure. The function f is defined for *all* x and tends to infinity as x approaches $\pm\infty$ (i.e., as P moves arbitrarily far to the right or left). It follows that f has an absolute minimum value, and it must occur at a critical point (see Figure 10). Taking the derivative:

$$f'(x) = \frac{x}{\sqrt{x^2 + h_1^2}} - \frac{L - x}{\sqrt{(L-x)^2 + h_2^2}} \qquad \boxed{1}$$

Since $f'(x)$ is defined for all x, critical points occur where $f'(x) = 0$. It is not necessary to solve for x because our goal is not to find critical points, but rather to show that $\theta_1 = \theta_2$ at the minimum. To do this, we set the derivative equal to 0 in Eq. (1) and rewrite as

$$\frac{x}{\sqrt{x^2 + h_1^2}} = \frac{L - x}{\sqrt{(L-x)^2 + h_2^2}} \qquad \boxed{2}$$

Note that the critical point x that satisfies Eq. (2) must lie between 0 and L because no $x < 0$ can satisfy this equation (otherwise, we would have a negative value on the left and a positive on the right) and no $x > L$ can satisfy this equation (for similar reasons). Since the critical point x lies in $[0, L]$ we can associate angles θ_1 and θ_2 with x as in Figure 9. We claim that $\theta_1 = \theta_2$. To see this, observe that with θ_1 and θ_2 as pictured, we have

$$\cos\theta_1 = \frac{x}{\sqrt{x^2 + h_1^2}} \quad \text{and} \quad \cos\theta_2 = \frac{L - x}{\sqrt{(L-x)^2 + h_2^2}}$$

Therefore, Eq. (2) implies that $\cos\theta_1 = \cos\theta_2$, and since θ_1 and θ_2 lie between 0 and $\frac{\pi}{2}$, we conclude that $\theta_1 = \theta_2$ as claimed. ■

CONCEPTUAL INSIGHT Often, a maximum or minimum at a critical point represents the best compromise between "competing factors." In Example 4, we maximized profit by finding the best compromise between raising the rent and keeping the apartment units occupied. In Example 5, our solution minimizes surface area by finding the best compromise between height and radius. In Example 2, the solution represents a compromise between the slower speed on the road that leads to Route 1 and the faster speed along Route 1. On the other hand, in Example 3, since there is no compromise, a solution occurs at an endpoint of the interval rather than at a critical point. The faster speed along the road yields a road straight to the city, avoiding Route 1 altogether.

4.7 SUMMARY

- There are usually three main steps in solving an applied optimization problem:

  Step 1. Choose variables.

  Determine which quantities are relevant, often by drawing a diagram, and assign appropriate variables.

  Step 2. Find the objective function and the interval.

  Restate as an optimization problem for a function f over an interval. If f depends on more than one variable, use a *constraint equation* to write f as a function of just one variable.

Step 3. Optimize the objective function.

- If the interval is open, f does not necessarily take on a minimum or maximum value. But if it does, these must occur at critical points within the interval. To determine if a min or max exists, analyze the behavior of f as x approaches the endpoints of the interval.

4.7 EXERCISES

Preliminary Questions

1. The problem is to find the right triangle of perimeter 10 whose area is as large as possible. What is the constraint equation relating the base b and height h of the triangle?

2. Describe a way of showing that a continuous function on an open interval (a, b) has a minimum value.

3. Is there a rectangle of area 100 of largest perimeter? Explain.

Exercises

1. Find the dimensions x and y of the rectangle of maximum area that can be formed using 3 m of wire.

(a) What is the constraint equation relating x and y?

(b) Find a formula for the area in terms of x alone.

(c) What is the interval of optimization? Is it open or closed?

(d) Solve the optimization problem.

2. Wire of length 12 m is divided into two pieces and each piece is bent into a square. How should this be done in order to minimize the sum of the areas of the two squares?

(a) Express the sum of the areas of the squares in terms of the lengths x and y of the two pieces.

(b) What is the constraint equation relating x and y?

(c) What is the interval of optimization? Is it open or closed?

(d) Solve the optimization problem.

3. A rectangular bird sanctuary is being created with one side along a straight riverbank. The remaining three sides are to be enclosed with a protective fence. If there are 12 km of fence available, find the dimension of the rectangle to maximize the area of the sanctuary.

4. The rectangular bird sanctuary with one side along a straight river is to be constructed so that it contains 8 km^2 of area. Find the dimensions of the rectangle to minimize the amount of fence necessary to enclose the remaining three sides.

5. Find two positive real numbers such that the sum of the first number squared and the second number is 48 and their product is a maximum.

6. Find two positive real numbers such that they sum to 108 and the product of the first times the square of the second is a maximum.

7. A wire of length 12 m is divided into two pieces and the pieces are bent into a square and a circle. How should this be done in order to minimize the sum of their areas?

8. Find the positive number x such that the sum of x and its reciprocal is as small as possible. Does this problem require optimization over an open interval or a closed interval?

9. Find two positive real numbers such that they add to 40 and their product is as large as possible.

10. Find two positive real numbers x and y such that they add to 120 and x^2y is as large as possible.

11. Find two positive real numbers x and y such that their product is 800 and $x + 2y$ is as small as possible.

12. A flexible tube of length 4 m is bent into an L-shape. Where should the bend be made to minimize the distance between the two ends?

13. Find the dimensions of the box with square base with

(a) Volume 12 and the minimal surface area.

(b) Surface area 20 and maximal volume.

14. A jewelry box with a square base is to be built with copper-plated sides, nickel-plated bottom and top, and a volume of 40 cm^3. If nickel plating costs \$2 per cm^2 and copper plating costs \$1 per cm^2, find the dimensions of the box to minimize the cost of the materials.

15. A rancher will use 600 m of fencing to build a corral in the shape of a semicircle on top of a rectangle (Figure 11). Find the dimensions that maximize the area of the corral.

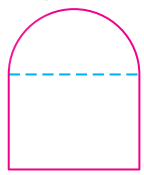


FIGURE 11

16. What is the maximum area of a rectangle inscribed in a right triangle with legs of length 3 and 4 as in Figure 12? The sides of the rectangle are parallel to the legs of the triangle.

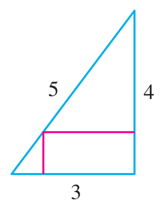


FIGURE 12

17. Find the dimensions of the rectangle of maximum area that can be inscribed in a circle of radius $r = 4$ (Figure 13).

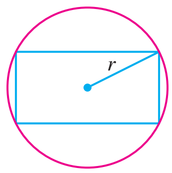


FIGURE 13

18. Find the dimensions x and y of the rectangle inscribed in a circle of radius r that maximizes the quantity xy^2.

19. In the setting of Examples 2 and 3, let r denote the speed along the road, and h denote the speed along the highway.

(a) Show that the travel-time function $T(x)$ has a critical point at

$$x = \frac{30}{\sqrt{(h/r)^2 - 1}}$$

and explain why this indicates that if $r \geq h$ there is no critical point.

(b) Explain why there cannot be a critical point at $x = 0$, but depending on the speeds, the critical point can be arbitrarily close to 0.

20. In the setting of Examples 2 and 3, replace 30 and 50 with general distances D and L, respectively. Also, let r denote the speed along the road, and h denote the speed along the highway. Show that the travel-time function $T(x)$ has a critical point at

$$x = \frac{D}{\sqrt{(h/r)^2 - 1}}.$$

21. In the article "Do Dogs Know Calculus?" the author Timothy Pennings explained how he noticed that when he threw a ball diagonally into Lake Michigan along a straight shoreline, his dog Elvis seemed to pick the optimal point in which to enter the water so as to minimize his time to reach the ball, as in Figure 14. He timed the dog and found Elvis could run at 6.4 m/s on the sand and swim at 0.91 m/s. If Tim stood at point A and threw the ball to a point B in the water, which was a perpendicular distance 10 m from point C on the shore, where C is a distance 15 m from where he stood, at what distance x from point C did Elvis enter the water if the dog effectively minimized his time to reach the ball?

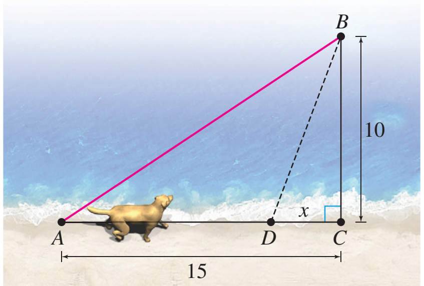


FIGURE 14

22. A four-wheel-drive vehicle is transporting an injured hiker to the hospital from a point that is 30 km from the nearest point on a straight road. The hospital is 50 km down that road from that nearest point. If the vehicle can drive at 30 kph over the terrain and at 120 kph on the road, how far down the road should the vehicle aim to reach the road to minimize the time it takes to reach the hospital?

23. Find the point on the line $y = x$ closest to the point $(1, 0)$. *Hint:* It is equivalent and easier to minimize the *square* of the distance.

24. Find the point P on the parabola $y = x^2$ closest to the point $(3, 0)$ (Figure 15).

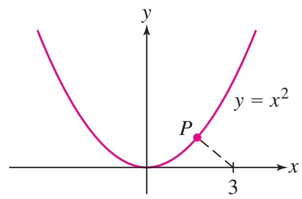


FIGURE 15

25. CAS Find a good numerical approximation to the coordinates of the point on the graph of $y = \ln x - x$ closest to the origin (Figure 16).

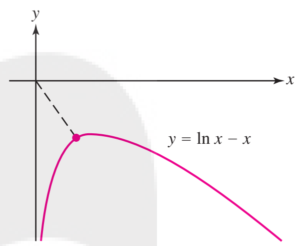


FIGURE 16

26. Problem of Tartaglia (1500–1557) Among all positive numbers a, b whose sum is 8, find those for which the product of the two numbers and their difference is largest.

27. Find the angle θ that maximizes the area of the isosceles triangle whose legs have length ℓ (Figure 17), using the fact the area is given by $A = \frac{1}{2}\ell^2 \sin\theta$.

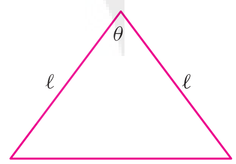


FIGURE 17

28. A right circular cone (Figure 18) has volume

$$V = \frac{\pi}{3}r^2 h$$

and surface area $S = \pi r\sqrt{r^2 + h^2}$. Find the dimensions of the cone with surface area 1 and maximal volume.

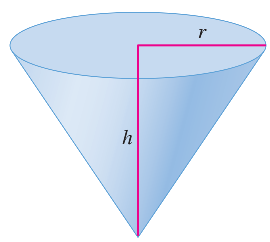


FIGURE 18

29. Find the area of the largest isosceles triangle that can be inscribed in a circle of radius 1 (Figure 19).

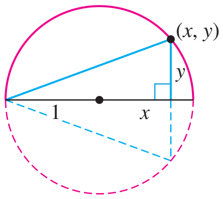


FIGURE 19

30. Find the radius and height of a cylindrical can of total surface area A whose volume is as large as possible. Does there exist a cylinder of surface area A and minimal total volume?

31. A poster of area 6000 cm^2 has blank margins of width 10 cm on the top and bottom and 6 cm on the sides. Find the dimensions that maximize the printed area.

32. According to postal regulations, a carton is classified as "oversized" if the sum of its height and girth (perimeter of its base) exceeds 108 in. Find the dimensions of a carton with a square base that is not oversized and has maximum volume.

33. Kepler's Wine Barrel Problem In his work *Nova stereometria doliorum vinariorum* (New Solid Geometry of a Wine Barrel), published in 1615, astronomer Johannes Kepler stated and solved the following problem: Find the dimensions of the cylinder of largest volume that can be inscribed in a sphere of radius R. *Hint:* Show that an inscribed cylinder has volume $2\pi x(R^2 - x^2)$, where x is one-half the height of the cylinder.

34. Find the angle θ that maximizes the area of the trapezoid with a base of length 4 and sides of length 2, as in Figure 20.

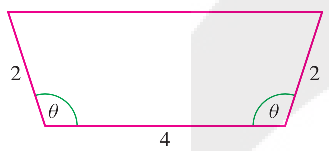


FIGURE 20

35. A landscape architect wishes to enclose a rectangular garden of area 1000 m^2 on one side by a brick wall costing \$90/m and on the other three sides by a metal fence costing \$30/m. Which dimensions minimize the total cost?

36. The amount of light reaching a point at a distance r from a light source A of intensity I_A is I_A/r^2. Suppose that a second light source B of intensity $I_B = 4I_A$ is located 10 m from A. Find the point on the segment joining A and B where the total amount of light is at a minimum.

37. Find the maximum area of a rectangle inscribed in the region bounded by the graph of $y = \dfrac{4-x}{2+x}$ and the axes (Figure 21).

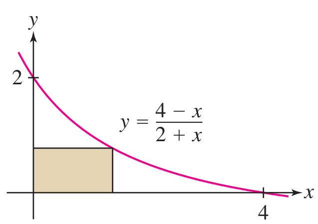


FIGURE 21

38. Find the maximum area of a triangle formed by the axes and a tangent line to the graph of $y = (x+1)^{-2}$ with $x > 0$.

39. Find the maximum area of a rectangle circumscribed around a rectangle of sides L and H. *Hint:* Express the area in terms of the angle θ (Figure 22).

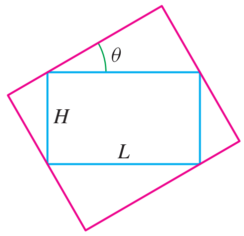


FIGURE 22

40. A contractor is engaged to build steps up the slope of a hill that has the shape of the graph of $y = \dfrac{x^2(120-x)}{6400}$ for $0 \le x \le 80$ with x in meters (Figure 23). What is the maximum vertical rise of a stair if each stair has a horizontal length of $\frac{1}{3}$ m?

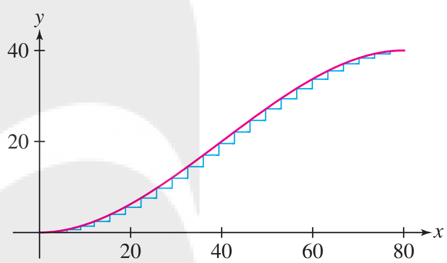


FIGURE 23

41. Find the equation of the line through $P = (4, 12)$ such that the triangle bounded by this line and the axes in the first quadrant has minimal area.

42. Let $P = (a, b)$ lie in the first quadrant. Find the slope of the line through P such that the triangle bounded by this line and the axes in the first quadrant has minimal area. Then show that P is the midpoint of the hypotenuse of this triangle.

43. Archimedes's Problem A spherical cap (Figure 24) of radius r and height h has volume $V = \pi h^2\left(r - \frac{1}{3}h\right)$ and surface area $S = 2\pi r h$. Prove that the hemisphere encloses the largest volume among all spherical caps of fixed surface area S.

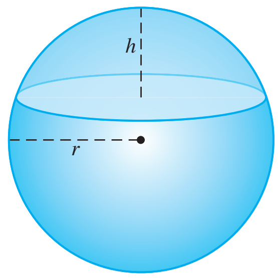


FIGURE 24

44. Find the isosceles triangle of smallest area (Figure 25) that circumscribes a circle of radius 1 (from Thomas Simpson's *The Doctrine and Application of Fluxions*, a calculus text that appeared in 1750).

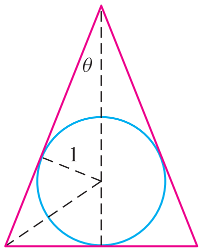


FIGURE 25

45. A box of volume 72 m^3 with a square bottom and no top is constructed out of two different materials. The cost of the bottom is \$40/m^2 and the cost of the sides is \$30/m^2. Find the dimensions of the box that minimize total cost.

46. Find the dimensions of a cylinder of volume 1 m^3 of minimal cost if the top and bottom are made of material that costs twice as much as the material for the side.

47. Your task is to design a rectangular industrial warehouse consisting of three separate spaces of equal size as in Figure 26. The wall materials cost \$500 per linear meter and your company allocates \$2,400,000 for that part of the project involving the walls.

(a) Which dimensions maximize the area of the warehouse?

(b) What is the area of each compartment in this case?

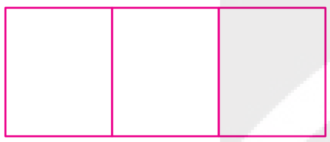


FIGURE 26

48. Suppose, in the previous exercise, that the warehouse consists of n separate spaces of equal size. Find a formula in terms of n for the maximum possible area of the warehouse.

49. According to a model developed by economists E. Heady and J. Pesek, if fertilizer made from N pounds of nitrogen and P lb of phosphate is used on an acre of farmland, then the yield of corn (in bushels per acre) is

$$Y = 7.5 + 0.6N + 0.7P - 0.001N^2 - 0.002P^2 + 0.001NP$$

A farmer intends to spend \$30/acre on fertilizer. If nitrogen costs 25 cents/lb and phosphate costs 20 cents/lb, which combination of N and P produces the highest yield of corn?

50. Experiments show that the quantities x of corn and y of soybean required to produce a hog of weight Q satisfy $Q = 0.5x^{1/2}y^{1/4}$. The unit of x, y, and Q is the cwt, an agricultural unit equal to 100 lb. Find the values of x and y that minimize the cost of a hog of weight $Q = 2.5$ cwt if corn costs \$3/cwt and soy costs \$7/cwt.

51. All units in a 100-unit apartment building are rented out when the monthly rent is set at $r = \$900$/month. Suppose that one unit becomes vacant with each \$10 increase in rent and that each occupied unit costs \$80/mon in maintenance. Which rent r maximizes monthly profit?

52. An 8-billion-bushel corn crop brings a price of \$2.40/bushel. A commodity broker uses the rule of thumb: If the crop is reduced by x percent, then the price increases by $10x$ cents. Which crop size results in maximum revenue and what is the price per bushel? *Hint:* Revenue is equal to price times crop size.

53. The monthly output of a Spanish light bulb factory is $P = 2LK^2$ (in millions), where L is the cost of labor and K is the cost of equipment (in millions of euros). The company needs to produce 1.7 million units per month. Which values of L and K would minimize the total cost $L + K$?

54. The rectangular plot in Figure 27 has size 100 m $\times$ 200 m. Pipe is to be laid from A to a point P on side BC and from there to C. The cost of laying pipe along the side of the plot is \$45/m and the cost through the plot is \$80/m (since it is underground).

(a) Let $f(x)$ be the total cost, where x is the distance from P to B. Determine $f(x)$, but note that f is discontinuous at $x = 0$ (when $x = 0$, the cost of the entire pipe is \$45/m).

(b) What is the most economical way to lay the pipe? What if the cost along the sides is \$65/m?

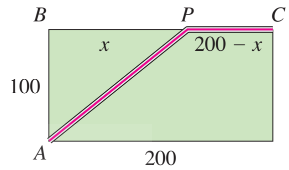


FIGURE 27

55. Brandon is on one side of a river that is 50 m wide and wants to reach a point 200 m downstream on the opposite side as quickly as possible by swimming diagonally across the river and then running the rest of the way. Find the best route if Brandon can swim at 1.5 m/s and run at 4 m/s.

56. Snell's Law When a light beam travels from a point A above a swimming pool to a point B below the water (Figure 28), it chooses the path that takes the *least time*. Let v_1 be the velocity of light in air and v_2 the velocity in water (it is known that $v_1 > v_2$). Prove Snell's Law of Refraction:

$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}$$

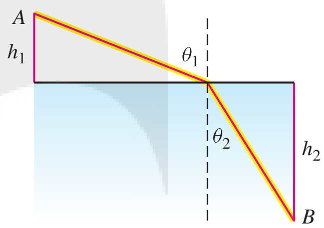


FIGURE 28

57. Vascular Branching A small blood vessel of radius r branches off at an angle θ from a larger vessel of radius R to supply blood along a path from A to B. According to Poiseuille's Law, the total resistance to blood flow is proportional to

$$T = \left(\frac{a - b\cot\theta}{R^4} + \frac{b\csc\theta}{r^4}\right)$$

where a and b are as in Figure 29. Show that the total resistance is minimized when $\cos\theta = (r/R)^4$.

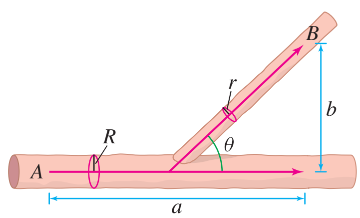


FIGURE 29

In Exercises 58–59, a box (with no top) is to be constructed from a piece of cardboard with sides of length A and B by cutting out squares of length h from the corners and folding up the sides (Figure 30).

58. Find the value of h that maximizes the volume of the box if $A = 15$ and $B = 24$. What are the dimensions of this box?

59. Which values of A and B maximize the volume of the box if $h = 10$ cm and $AB = 900$ cm^2?

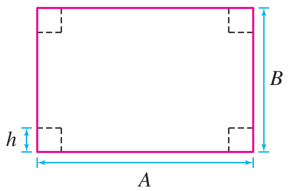


FIGURE 30

60. Which value of h maximizes the volume of the box if $A = B$?

61. Given n numbers $x_1, \dots, x_n$, find the value of x minimizing the sum of the squares:

$$(x - x_1)^2 + (x - x_2)^2 + \cdots + (x - x_n)^2$$

First, solve for $n = 2, 3$ and then try it for arbitrary n.

62. A billboard of height b is mounted on the side of a building with its bottom edge at a distance h from the street as in Figure 31. At what distance x should an observer stand from the wall to maximize the angle of observation θ?

63. Solve Exercise 62 again using geometry rather than calculus. There is a unique circle passing through points B and C that is tangent to the street. Let R be the point of tangency. Note that the two angles labeled ψ in Figure 31 are equal because they subtend equal arcs on the circle.

(a) Show that the maximum value of θ is $\theta = \psi$. *Hint:* Show that $\psi = \theta + \angle PBA$, where A is the intersection of the circle with PC.

(b) Prove that this agrees with the answer to Exercise 62.

(c) Show that $\angle QRB = \angle RCQ$ for the maximal angle ψ.

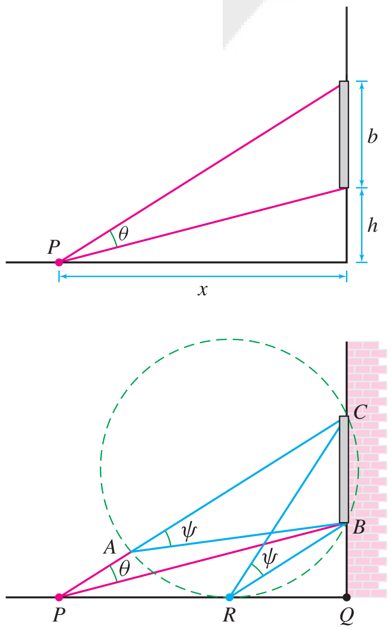


FIGURE 31

64. Optimal Delivery Schedule A gas station sells Q gallons of gasoline per year, which is delivered N times per year in equal shipments of Q/N gallons. The cost of each delivery is d dollars and the yearly storage costs are sQT, where T is the length of time (a fraction of a year) between shipments and s is a constant. Show that costs are minimized for $N = \sqrt{sQ/d}$. (*Hint:* $T = 1/N$.) Find the optimal number of deliveries if $Q = 2$ million gal, $d = \$8000$, and $s = 30$ cents/gal-year. Your answer should be a whole number, so compare costs for the two integer values of N nearest the optimal value.

65. Victor Klee's Endpoint Maximum Problem Given 40 m of straight fence, your goal is to build a rectangular enclosure using 80 additional meters of fence that encompasses the greatest area. Let $A(x)$ be the area of the enclosure, with x as in Figure 32.

(a) Find the maximum value of $A(x)$.

(b) Which interval of x values is relevant to our problem? Find the maximum value of $A(x)$ on this interval.

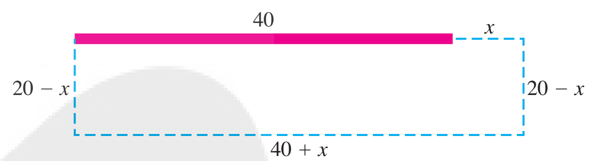


FIGURE 32

66. Let (a, b) be a fixed point in the first quadrant and let $S(d)$ be the sum of the distances from $(d, 0)$ to the points $(0, 0)$, (a, b), and $(a, -b)$.

(a) Find the value of d for which $S(d)$ is minimal. The answer depends on whether $b < \sqrt{3}a$ or $b \ge \sqrt{3}a$. *Hint:* Show that $d = 0$ when $b \ge \sqrt{3}a$.

(b) GU Let $a = 1$. Plot S for $b = 0.5, \sqrt{3}, 3$ and describe the position of the minimum.

67. The force F (in Newtons) required to move a box of mass m kg in motion by pulling on an attached rope (Figure 33) is

$$F(\theta) = \frac{fmg}{\cos\theta + f\sin\theta}$$

where θ is the angle between the rope and the horizontal, f is the coefficient of static friction, and $g = 9.8$ m/s^2. Find the angle θ that minimizes the required force F, assuming $f = 0.4$. *Hint:* Find the maximum value of $\cos\theta + f\sin\theta$.

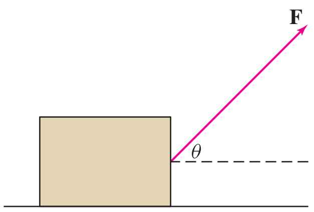


FIGURE 33

68. In the setting of Exercise 67, show that for any f the minimal force required is proportional to $1/\sqrt{1 + f^2}$.

69. Bird Migration Ornithologists have found that the power (in joules per second) consumed by a certain pigeon flying at velocity v m/s is described well by the function $P(v) = 17v^{-1} + 10^{-3}v^3$ joules/s. Assume that the pigeon can store 5×10^4 joules of usable energy as body fat.

(a) Show that at velocity v, a pigeon can fly a total distance of $D(v) = (5 \times 10^4)v/P(v)$ if it uses all of its stored energy.

(b) Find the velocity v_p that *minimizes* P.

(c) Migrating birds are smart enough to fly at the velocity that maximizes distance traveled rather than minimizes power consumption. Show that the velocity v_d which maximizes $D(v)$ satisfies $P'(v_d) = P(v_d)/v_d$. Show that v_d is obtained graphically as the velocity coordinate of the point where a line through the origin is tangent to the graph of P (Figure 34).

(d) Find v_d and the maximum distance $D(v_d)$.

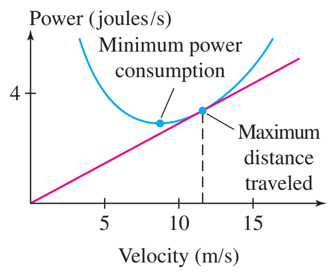


FIGURE 34

70. The problem is to put a "roof" of side s on an attic room of height h and width b. Find the smallest length s for which this is possible if $b = 27$ and $h = 8$ (Figure 35).

71. Redo Exercise 70 for arbitrary b and h.

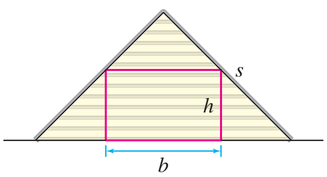


FIGURE 35

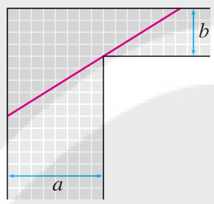


FIGURE 36

72. Find the maximum length of a pole that can be carried horizontally around a corner joining corridors of widths $a = 24$ and $b = 3$ (Figure 36).

73. Redo Exercise 72 for arbitrary widths a and b.

74. Find the minimum length ℓ of a beam that can clear a fence of height h and touch a wall located b ft behind the fence (Figure 37).

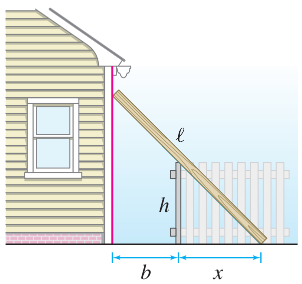


FIGURE 37

75. A basketball player stands d feet from the basket. Let h and α be as in Figure 38. Using physics, one can show that if the player releases the ball at an angle θ, then the initial velocity required to make the ball go through the basket satisfies

$$v^2 = \frac{16d}{\cos^2\theta(\tan\theta - \tan\alpha)}$$

(a) Explain why this formula is meaningful only for $\alpha < \theta < \frac{\pi}{2}$. Why does v approach infinity at the endpoints of this interval?

(b) GU Take $\alpha = \frac{\pi}{6}$ and plot v^2 as a function of θ for $\frac{\pi}{6} < \theta < \frac{\pi}{2}$. Verify that the minimum occurs at $\theta = \frac{\pi}{3}$.

(c) Set $F(\theta) = \cos^2\theta(\tan\theta - \tan\alpha)$. Explain why v is minimized for θ such that $F(\theta)$ is maximized.

(d) Verify that $F'(\theta) = \cos(\alpha - 2\theta)\sec\alpha$ (you will need to use the addition formula for cosine) and show that the maximum value of F on $\left[\alpha, \frac{\pi}{2}\right]$ occurs at $\theta_0 = \frac{\alpha}{2} + \frac{\pi}{4}$.

(e) For a given α, the optimal angle for shooting the basket is θ_0 because it minimizes v^2 and therefore minimizes the energy required to make the shot (energy is proportional to v^2). Show that the velocity v_{opt} at the optimal angle θ_0 satisfies

$$v_{\text{opt}}^2 = \frac{32d\cos\alpha}{1 - \sin\alpha} = \frac{32\,d^2}{-h + \sqrt{d^2 + h^2}}$$

(f) GU Show with a graph that for fixed d (say, $d = 15$ ft, the distance of a free throw), v_{opt}^2 is an increasing function of h. Use this to explain why taller players have an advantage and why it can help to jump while shooting.

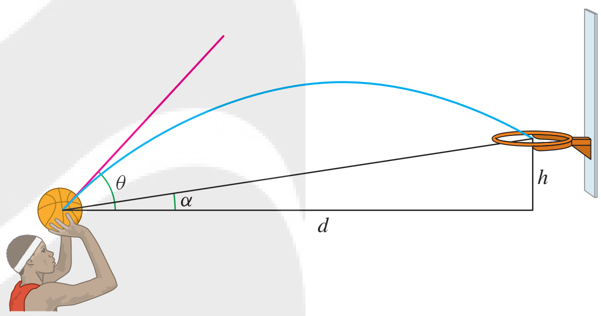


FIGURE 38

76. Three towns A, B, and C are to be joined by an underground fiber cable as illustrated in Figure 39(A). Assume that C is located directly below the midpoint of $\overline{AB}$. Find the junction point P that minimizes the total amount of cable used.

(a) First show that P must lie directly above C. *Hint:* Use the result of Example 7 to show that if the junction is placed at point Q in Figure 39(B), then we can reduce the cable length by moving Q horizontally over to the point P lying above C.

(b) With x as in Figure 39(A), let $f(x)$ be the total length of cable used. Show that f has a unique critical point c. Compute c and show that $0 \le c \le L$ if and only if $D \le 2\sqrt{3}\,L$.

(c) Find the minimum of f on $[0, L]$ in two cases: $D = 2$, $L = 4$ and $D = 8$, $L = 2$.

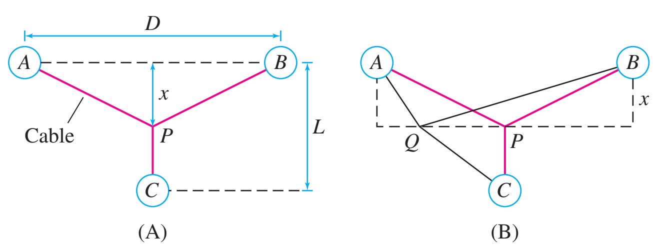


FIGURE 39

Further Insights and Challenges

77. Tom and Ali drive along a highway represented by the graph of f in Figure 40. During the trip, Ali views a billboard represented by the segment $\overline{BC}$ along the y-axis. Let Q be the y-intercept of the tangent line to $y = f(x)$. Show that θ is maximized at the value of x for which the angles $\angle QPB$ and $\angle QCP$ are equal. This generalizes Exercise 63 (c) [which corresponds to the case $f(x) = 0$]. *Hints:*

(a) Show that $d\theta/dx$ is equal to

$$(b-c)\cdot\frac{(x^2+(xf'(x))^2)-(b-(f(x)-xf'(x)))(c-(f(x)-xf'(x)))}{(x^2+(b-f(x))^2)(x^2+(c-f(x))^2)}$$

(b) Show that the y-coordinate of Q is $f(x) - xf'(x)$.

(c) Show that the condition $d\theta/dx = 0$ is equivalent to

$$PQ^2 = BQ \cdot CQ$$

(d) Conclude that $\triangle QPB$ and $\triangle QCP$ are similar triangles.

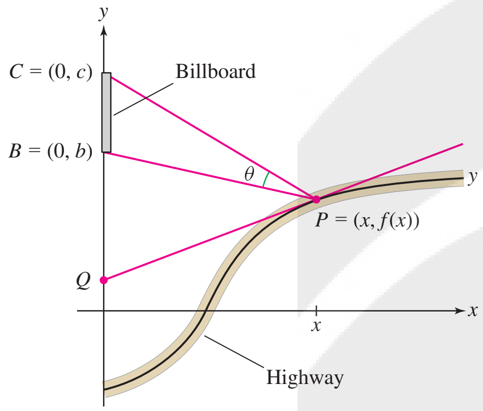


FIGURE 40

Seismic Prospecting *Exercises 78–80 are concerned with determining the thickness d of a layer of soil that lies on top of a rock formation. Geologists send two sound pulses from point A to point D separated by a distance s. The first pulse travels directly from A to D along the surface of the earth. The second pulse travels down to the rock formation, then along its surface, and then back up to D (path $ABCD$), as in Figure 41. The pulse travels with velocity v_1 in the soil and v_2 in the rock.*

78. (a) Show that the time required for the first pulse to travel from A to D is $t_1 = s/v_1$.

(b) Show that the time required for the second pulse is

$$t_2 = \frac{2d}{v_1}\sec\theta + \frac{s - 2d\tan\theta}{v_2}$$

provided that

$$\tan\theta \le \frac{s}{2d} \qquad \boxed{3}$$

(*Note:* If this inequality is not satisfied, then point B does not lie to the left of C.)

(c) Show that t_2 is minimized when $\sin\theta = v_1/v_2$.

79. In this exercise, assume that $v_2/v_1 \ge \sqrt{1 + 4(d/s)^2}$.

(a) Show that inequality (3) holds if $\sin\theta = v_1/v_2$.

(b) Show that the minimal time for the second pulse is

$$t_2 = \frac{2d}{v_1}(1-k^2)^{1/2} + \frac{s}{v_2}$$

where $k = v_1/v_2$.

(c) Conclude that $\dfrac{t_2}{t_1} = \dfrac{2d(1-k^2)^{1/2}}{s} + k$.

80. Continue with the assumption of the previous exercise.

(a) Find the thickness of the soil layer, assuming that $v_1 = 0.7v_2$, $t_2/t_1 = 1.3$, and $s = 400$ m.

(b) The times t_1 and t_2 are measured experimentally. The equation in Exercise 79(c) shows that t_2/t_1 is a linear function of $1/s$. What might you conclude if experiments were formed for several values of s and the points $(1/s, t_2/t_1)$ did *not* lie on a straight line?

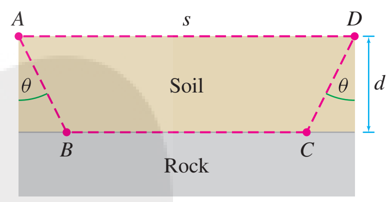


FIGURE 41

81. In this exercise, we use Figure 42 to prove Heron's principle of Example 7 without calculus. By definition, C is the reflection of B across the line $\overline{MN}$ (so that $\overline{BC}$ is perpendicular to $\overline{MN}$ and $BN = CN$). Let P be the intersection of $\overline{AC}$ and $\overline{MN}$. Use geometry to justify the following:

(a) $\triangle PNB$ and $\triangle PNC$ are congruent and $\theta_1 = \theta_2$.

(b) The paths APB and APC have equal length.

(c) Similarly, AQB and AQC have equal length.

(d) The path APC is shorter than AQC for all $Q \neq P$.

Conclude that the shortest path AQB occurs for $Q = P$.

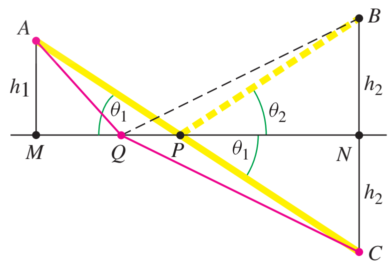


FIGURE 42

82. A jewelry designer plans to incorporate a component made of gold in the shape of a frustum of a cone of height 1 cm and fixed lower radius r (Figure 43). The upper radius x can take on any value between 0 and r. Note that $x = 0$ and $x = r$ correspond to a cone and cylinder, respectively. As a function of x, the surface area (not including the top and bottom) is $S(x) = \pi s(r + x)$, where s is the *slant height* as indicated in the figure. Which value of x yields the least expensive design [the minimum value of $S(x)$ for $0 \le x \le r$]?

(a) Show that $S(x) = \pi(r + x)\sqrt{1 + (r - x)^2}$.

(b) Show that if $r < \sqrt{2}$, then S is an increasing function. Conclude that the cone ($x = 0$) has minimal area in this case.

(c) Assume that $r > \sqrt{2}$. Show that S has two critical points $x_1 < x_2$ in $(0, r)$, and that $S(x_1)$ is a local maximum, and $S(x_2)$ is a local minimum.

(d) Conclude that the minimum occurs at $x = 0$ or x_2.

(e) Find the minimum in the cases $r = 1.5$ and $r = 2$.

(f) Challenge: Let $c = \sqrt{(5 + 3\sqrt{3})/4} \approx 1.597$. Prove that the minimum occurs at $x = 0$ (cone) if $\sqrt{2} < r < c$, but the minimum occurs at $x = x_2$ if $r > c$.

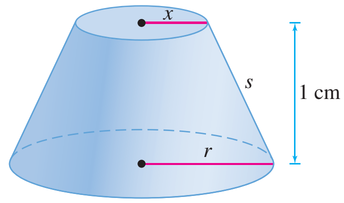


FIGURE 43 Frustum of height 1 cm.

4.8 Newton's Method

REMINDER A "zero" or "root" of a function f is a solution of the equation $f(x) = 0$.

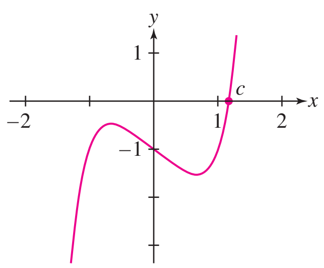
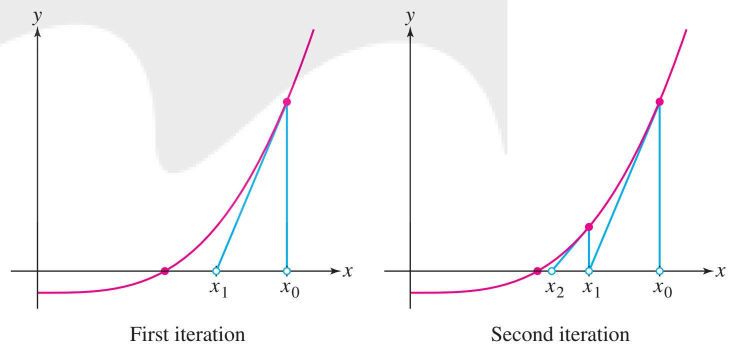


FIGURE 1 Graph of $y = x^5 - x - 1$. With Newton's Method, we can approximate the root c as accurately as we like.

Newton's Method is a procedure for finding numerical approximations to zeros of functions. Numerical approximations are important because it is often impossible to find the zeros exactly. For example, the polynomial $f(x) = x^5 - x - 1$ has one real root c (see Figure 1), but we can prove, using an advanced branch of mathematics called *Galois Theory*, that there is no algebraic formula for this root. In this section, using Newton's Method, we show that $c \approx 1.1673$, and we show that we can compute c to any desired degree of accuracy with enough computation.

In Newton's Method, we begin by choosing a number x_0, which we believe is close to a root of the equation $f(x) = 0$. This starting value x_0 is called the **initial guess**. Newton's Method then produces a sequence $x_0, x_1, x_2, x_3 \ldots$ of successive approximations that, in favorable situations, converge to a root.

Figure 2 illustrates the procedure. Given an initial guess x_0, we draw the tangent line to the graph at $(x_0, f(x_0))$. The approximation x_1 is defined as the x-coordinate of the point where the tangent line intersects the x-axis. To produce the second approximation x_2 (also called the second iterate), we apply this procedure to x_1. Then, repeatedly applying this procedure, we produce the sequence of approximations $x_0, x_1, x_2, x_3 \ldots$.



DF **FIGURE 2** The sequence produced by iteration converges to a root.

Let's derive a formula for x_1. The tangent line at $(x_0, f(x_0))$ has equation

$$y = f(x_0) + f'(x_0)(x - x_0)$$

The tangent line crosses the x-axis at x_1, where $y = 0$, that is, where

$$f(x_0) + f'(x_0)(x_1 - x_0) = 0$$

To solve for x_1, we first divide by $f'(x_0)$ (as long as it is not zero) to obtain $x_1 - x_0 = -f(x_0)/f'(x_0)$, and therefore,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

The second iterate x_2 is obtained by applying this formula to x_1 instead of x_0:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

and so on. Notice in Figure 2 that x_1 is closer to the root than x_0 is and that x_2 is closer still. This is typical: The successive approximations usually converge to the actual root. However, there are cases where Newton's Method fails (see Figure 4).

Newton's Method To approximate a root of $f(x) = 0$:

Step 1. **Choose an initial guess x_0 (close to the desired root if possible).**

Step 2. **Generate successive approximations $x_1, x_2, \ldots,$ where**

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad \boxed{1}$$

Newton's Method is an example of an iterative procedure. To "iterate" means to repeat, and in Newton's Method, we use Eq. (1) repeatedly to produce the sequence of approximations.

EXAMPLE 1 Calculate the first five approximations $x_1, \ldots, x_5$ to a root of $f(x) = x^5 - x - 1$ using the initial guess $x_0 = 1$.

Solution We have $f'(x) = 5x^4 - 1$. Therefore,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{x_0^5 - x_0 - 1}{5x_0^4 - 1}$$

We compute the first two approximations as follows:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{1^5 - 1 - 1}{5(1)^4 - 1} = 1.25$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.25 - \frac{1.25^5 - 1.25 - 1}{5(1.25)^4 - 1} \approx 1.178459$$

Continuing, rounding to six decimal places at each stage, we obtain $x_3 \approx 1.167547$, $x_4 \approx 1.167304$, and $x_5 \approx 1.167304$. This suggests that, accurate to six decimal places, 1.167304 is a root of $f(x) = x^5 - x - 1$. ■

We can check our approximation; evaluating $x^5 - x - 1$ at $x = 1.167304$, we obtain 0.00000018 (to eight decimal places), verifying that we have a good approximation to a root of $f(x) = x^5 - x - 1$.

How Many Iterations Are Required?

How many iterations of Newton's Method are required to approximate a root to within a given accuracy? There is no definitive answer, but in practice, it is usually safe to assume that if x_n and x_{n+1} agree to m decimal places, then the approximation x_n is correct to these m places.

EXAMPLE 2 GU Let c be the smallest positive solution of $\sin 3x = \cos x$.

(a) Use a computer-generated graph to choose an initial guess x_0 for c.

(b) Use Newton's Method to approximate c to within an error of at most 10^{-6}.

Solution

(a) A solution of $\sin 3x = \cos x$ is a zero of the function $f(x) = \sin 3x - \cos x$. Figure 3 shows that the smallest positive zero is approximately halfway between 0 and $\frac{\pi}{4}$. Because $\frac{\pi}{4} \approx 0.785$, a good initial guess is $x_0 = 0.4$.

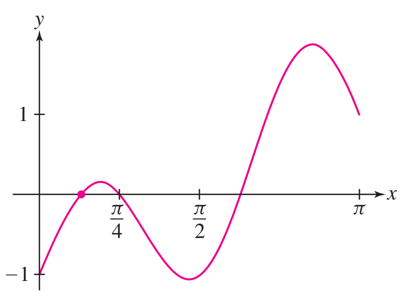


FIGURE 3 Graph of $f(x) = \sin 3x - \cos x$.

(b) Since $f'(x) = 3\cos 3x + \sin x$, Eq. (1) yields the formula

$$x_{n+1} = x_n - \frac{\sin 3x_n - \cos x_n}{3\cos 3x_n + \sin x_n}$$

With $x_0 = 0.4$ as the initial guess, the first four iterates are

$$x_1 \approx 0.3925647447$$
$$x_2 \approx 0.3926990382$$
$$x_3 \approx 0.3926990816987196$$
$$x_4 \approx 0.3926990816987241$$

Stopping here, we can be fairly confident that x_4 approximates the smallest positive root c to at least 12 places. In fact, $c = \frac{\pi}{8}$ and x_4 is accurate to 16 places. ■

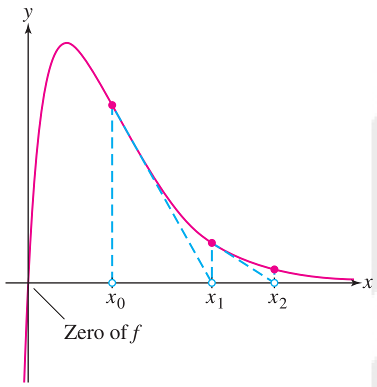
There is no single "correct" initial guess. In Example 2, we chose $x_0 = 0.4$, but another possible choice is $x_0 = 0$, leading to the sequence

$$x_1 \approx 0.3333333333$$
$$x_2 \approx 0.3864547725$$
$$x_3 \approx 0.3926082513$$
$$x_4 \approx 0.3926990816$$

You can check, however, that $x_0 = 1$ yields a sequence converging to $\frac{\pi}{4}$, which is the second positive solution of $\sin 3x = \cos x$.

Which Root Does Newton's Method Compute?

Sometimes, Newton's Method computes no root at all. In Figure 4, the iterates diverge to infinity. In practice, however, Newton's Method usually converges quickly, and if a particular choice of x_0 does not lead to a root, the best strategy is to try a different initial guess, consulting a graph if possible. If $f(x) = 0$ has more than one root, different initial guesses x_0 may lead to different roots.



DF **FIGURE 4** Function has only one zero but the sequence of Newton iterates goes off to infinity.

EXAMPLE 3 Figure 5 shows that $f(x) = x^4 - 6x^2 + x + 5$ has four real roots.

(a) Show that with $x_0 = 0$, Newton's Method converges to the root near -2.

(b) Show that with $x_0 = -1$, Newton's Method converges to the root near -1.

Solution We have $f'(x) = 4x^3 - 12x + 1$ and

$$x_{n+1} = x_n - \frac{x_n^4 - 6x_n^2 + x_n + 5}{4x_n^3 - 12x_n + 1} = \frac{3x_n^4 - 6x_n^2 - 5}{4x_n^3 - 12x_n + 1}$$

(a) On the basis of Table 1, we can be confident that when $x_0 = 0$, Newton's Method converges to a root near -2.3. Notice in Figure 5 that this is not the closest root to x_0.

(b) Table 2 suggests that with $x_0 = -1$, Newton's Method converges to the root near -0.9. ■

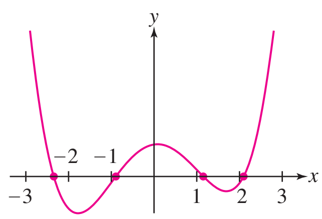


FIGURE 5 Graph of $f(x) = x^4 - 6x^2 + x + 5$.

TABLE 1

x_0	0
x_1	−5
x_2	−3.9179954
x_3	−3.1669480
x_4	−2.6871270
x_5	−2.4363303
x_6	−2.3572979
x_7	−2.3495000

TABLE 2

x_0	−1
x_1	−0.8888888888
x_2	−0.8882866140
x_3	−0.88828656234358
x_4	−0.888286562343575

EXAMPLE 4 **Approximating $\sqrt{5}$** We know that the solutions to $x^2 - 5 = 0$ are $x = \pm\sqrt{5}$. We can use Newton's method to obtain approximations to these values. Approximate $\sqrt{5}$ using an initial guess $x_0 = 2$.

Solution We have $f'(x) = 2x$. Therefore,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{x_0^2 - 5}{2x_0}$$

We compute the successive approximations as follows:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{2^2 - 5}{2 \cdot 2} = 2.25$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.25 - \frac{2.25^2 - 5}{2 \cdot 2.25} \approx 2.23611$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.23611 - \frac{2.23611^2 - 5}{2 \cdot 2.23611} \approx 2.23606797789$$

Therefore, $\sqrt{5} \approx 2.23606797789$.

A calculator computation of $\sqrt{5}$ yields

$$\sqrt{5} = 2.23606797750\ldots$$

Observe that x_3 is accurate to within an error of less than 10^{-9}. This is impressive accuracy for just three iterations of Newton's Method. ■

4.8 SUMMARY

- Newton's Method: To find a sequence of numerical approximations to a root of f, begin with an initial guess x_0. Then construct the sequence $x_0, x_1, x_2, \ldots$ using the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

  You should choose the initial guess x_0 as close as possible to a root, possibly by referring to a graph. In favorable cases, the sequence converges rapidly to a root.
- If x_n and x_{n+1} agree to m decimal places, it is usually safe to assume that x_n agrees with a root to m decimal places.

4.8 EXERCISES

Preliminary Questions

1. How many iterations of Newton's Method are required to compute a root if f is a linear function?

2. What happens in Newton's Method if your initial guess happens to be a zero of f?

3. What happens in Newton's Method if your initial guess happens to be a local min or max of f?

4. Is the following a reasonable description of Newton's Method: "A root of the equation of the tangent line to the graph of f is used as an approximation to a root of f itself"? Explain.

Exercises

In this exercise set, all approximations should be carried out using Newton's Method.

In Exercises 1–6, apply Newton's Method to f and initial guess x_0 to calculate x_1, x_2, x_3.

1. $f(x) = x^2 - 6, \quad x_0 = 2$

2. $f(x) = x^2 - 3x + 1, \quad x_0 = 3$

3. $f(x) = x^3 - 10, \quad x_0 = 2$

4. $f(x) = x^3 + x + 1, \quad x_0 = -1$

5. $f(x) = \cos x - 4x, \quad x_0 = 1$

6. $f(x) = 1 - x \sin x, \quad x_0 = 7$

7. Use Figure 6 to choose an initial guess x_0 to the unique real root of $x^3 + 2x + 5 = 0$ and compute the first three Newton iterates.

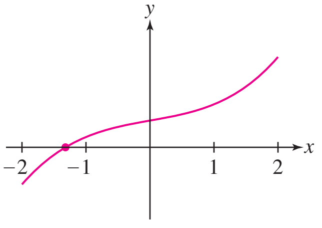


FIGURE 6 Graph of $y = x^3 + 2x + 5$.

8. Approximate a solution of $\sin x = \cos 2x$ in the interval $\left[0, \frac{\pi}{2}\right]$ to three decimal places. Then find the exact solution and compare with your approximation.

9. Approximate both solutions of $e^x = 5x$ to three decimal places (Figure 7).

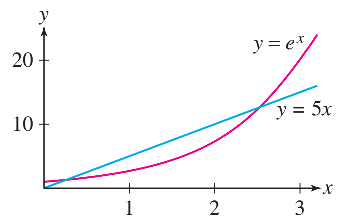


FIGURE 7 Graphs of $y = e^x$ and $y = 5x$.

10. The first positive solution of $\sin x = 0$ is $x = \pi$. Use Newton's Method to calculate π to four decimal places.

In Exercises 11–14, approximate to three decimal places using Newton's Method and compare with the value from a calculator.

11. $\sqrt{11}$ **12.** $5^{1/3}$ **13.** $2^{7/3}$ **14.** $3^{-1/4}$

15. Approximate the largest positive root of $f(x) = x^4 - 6x^2 + x + 5$ to within an error of at most 10^{-4}. Refer to Figure 5.

GU *In Exercises 16–21, approximate the value specified to three decimal places using Newton's Method. Use a plot to choose an initial guess.*

16. Largest positive root of $f(x) = x^3 - 5x + 1$

17. Negative root of $f(x) = x^5 - 20x + 10$

18. Positive solution of $\sin\theta = 0.8\theta$

19. Positive solution of $2\tan^{-1} x = x$

20. The least positive solution of $x \cos x = 10$

21. Solution of $\ln(x + 4) = x$

22. Let x_1, x_2 be the estimates to a root obtained by applying Newton's Method with $x_0 = 1$ to the function graphed in Figure 8. Estimate the numerical values of x_1 and x_2, and draw the tangent lines used to obtain them.

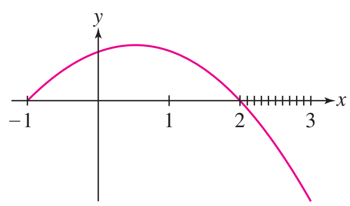


FIGURE 8

23. GU Find the smallest positive value of x at which $y = x$ and $y = \tan x$ intersect. *Hint:* Draw a plot.

24. In 1535, the mathematician Antonio Fior challenged his rival Niccolo Tartaglia to solve this problem: A tree stands 12 *braccia* high; it is broken into two parts at such a point that the height of the part left standing is the cube root of the length of the part cut away. What is the height of the part left standing? Show that this is equivalent to solving $x^3 + x = 12$ and finding the height to three decimal places. Tartaglia, who had discovered the secret of solving the cubic equation, was able to determine the exact answer:

$$x = \frac{\left(\sqrt[3]{\sqrt{2919} + 54} - \sqrt[3]{\sqrt{2919} - 54}\right)}{\sqrt[3]{9}}$$

25. Find (to two decimal places) the coordinates of the point P in Figure 9 where the tangent line to $y = \cos x$ passes through the origin.

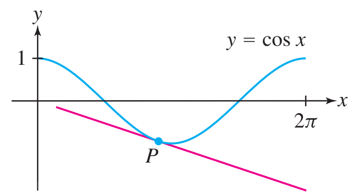


FIGURE 9

Newton's Method is often used to determine interest rates in financial calculations. In Exercises 26–28, r denotes a yearly interest rate expressed as a decimal (rather than as a percent).

26. If P dollars are deposited every month in an account earning interest at the yearly rate r, then the value S of the account after N years is

$$S = P\left(\frac{b^{12N+1} - b}{b - 1}\right), \qquad \text{where } b = 1 + \frac{r}{12}$$

You have decided to deposit $P = \$100$ per month.

(a) Determine S after 5 years if $r = 0.07$ (i.e., 7%).

(b) Show that to save \$10,000 after 5 years, you must earn interest at a rate r determined by the equation $b^{61} - 101b + 100 = 0$. Use Newton's Method to solve for b. Then find r. Note that $b = 1$ is a root, but you want the root satisfying $b > 1$.

27. If you borrow L dollars for N years at a yearly interest rate r, your monthly payment of P dollars is calculated using the equation

$$L = P\left(\frac{1 - b^{-12N}}{b - 1}\right), \qquad \text{where } b = 1 + \frac{r}{12}$$

(a) Find P if $L = \$5000$, $N = 3$, and $r = 0.08$ (8%).

(b) You are offered a loan of $L = \$5000$ to be paid back over 3 years with monthly payments of $P = \$200$. Use Newton's Method to compute b and find the implied interest rate r of this loan. *Hint:* Show that

$$(L/P)b^{12N+1} - (1 + L/P)b^{12N} + 1 = 0$$

28. If you deposit P dollars in a retirement fund every year for N years with the intention of then withdrawing Q dollars per year for M years, you must earn interest at a rate r satisfying

$$P(b^N - 1) = Q(1 - b^{-M}), \qquad \text{where } b = 1 + r$$

Assume that \$2000 is deposited each year for 30 years and the goal is to withdraw \$10,000 per year for 25 years. Use Newton's Method to compute b and then find r. Note that $b = 1$ is a root, but you want the root satisfying $b > 1$.

29. There is no simple formula for the position at time t of a planet P in its orbit (an ellipse) around the sun. Introduce the auxiliary circle and angle θ in Figure 10 (note that P determines θ because it is the central angle of point B on the circle). Let $a = OA$ and $e = OS/OA$ (the eccentricity of the orbit).

(a) Show that sector BSA has area $(a^2/2)(\theta - e\sin\theta)$.

(b) By Kepler's Second Law, the area of sector BSA is proportional to the time t elapsed since the planet passed point A, and because the circle has area πa^2, BSA has area $(\pi a^2)(t/T)$, where T is the period of the orbit. Deduce **Kepler's Equation**:

$$\frac{2\pi t}{T} = \theta - e\sin\theta$$

(c) The eccentricity of Mercury's orbit is approximately $e = 0.2$. Use Newton's Method to find θ after a quarter of Mercury's year has elapsed ($t = T/4$). Convert θ to degrees. Has Mercury covered more than a quarter of its orbit at $t = T/4$?

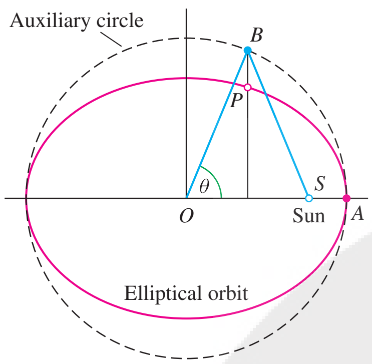


FIGURE 10

30. The roots of $f(x) = \frac{1}{3}x^3 - 4x + 1$ to three decimal places are -3.583, 0.251, and 3.332 (Figure 11). Determine the root to which Newton's Method converges for the initial choices $x_0 = 1.85$, 1.7, and 1.55. The answer shows that a small change in x_0 can have a significant effect on the outcome of Newton's Method.

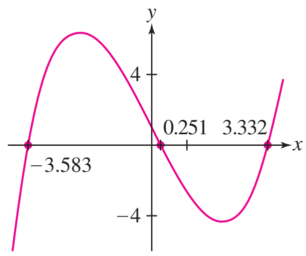


FIGURE 11 Graph of $f(x) = \frac{1}{3}x^3 - 4x + 1$.

31. Let $f(x) = xe^{-x}$.

(a) Show that a sequence obtained by applying Newton's Method to f satisfies $x_{n+1} = \frac{x_n^2}{x_n - 1}$.

(b) CAS Compute $x_1, x_2, \ldots, x_{10}$ separately with $x_0 = 0.8$ and $x_0 = 5$. Discuss what appears to be happening in each case. (*Note:* the only root of f is at $x = 0$.)

32. Let $f(x) = xe^{2x}$.

(a) Show that a sequence obtained by applying Newton's Method to f satisfies $x_{n+1} = \frac{2x_n^2}{2x_n + 1}$.

(b) CAS Compute $x_1, x_2, \ldots, x_{10}$ separately with $x_0 = -3$ and $x_0 = 7$. Discuss what appears to be happening in each case. (*Note:* the only root of f is at $x = 0$.)

33. What happens when you apply Newton's Method to find a zero of $f(x) = x^{1/3}$? Note that $x = 0$ is the only zero.

34. What happens when you apply Newton's Method to the equation $x^3 - 20x = 0$ with the unlucky initial guess $x_0 = 2$?

Further Insights and Challenges

35. Newton's Method can be used to compute reciprocals without performing division. Let $c > 0$ and set $f(x) = x^{-1} - c$.

(a) Show that $x - (f(x)/f'(x)) = 2x - cx^2$.

(b) Calculate the first three iterates of Newton's Method with $c = 10.3$ and the two initial guesses $x_0 = 0.1$ and $x_0 = 0.5$.

(c) Explain graphically why $x_0 = 0.5$ does not yield a sequence converging to $1/10.3$.

In Exercises 36 and 37, consider a metal rod of length L fastened at both ends. If you cut the rod and weld on an additional segment of length m, leaving the ends fixed, the rod will bow up into a circular arc of radius R (unknown), as indicated in Figure 12.

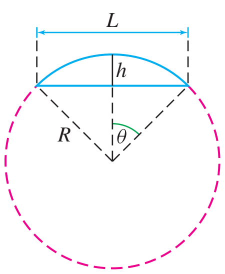


FIGURE 12 The bold circular arc has length $L + m$.

36. Let h be the maximum vertical displacement of the rod.

(a) Show that $L = 2R\sin\theta$ and conclude that

$$h = \frac{L(1 - \cos\theta)}{2\sin\theta}$$

(b) Show that $L + m = 2R\theta$ and then prove

$$\frac{\sin\theta}{\theta} = \frac{L}{L + m} \tag{2}$$

37. Let $L = 3$ and $m = 1$. Apply Newton's Method to Eq. (2) to estimate θ, and use this to estimate h.

38. Quadratic Convergence to Square Roots Let $f(x) = x^2 - c$ and let $e_n = x_n - \sqrt{c}$ be the error in x_n.

(a) Show that $x_{n+1} = \frac{1}{2}(x_n + c/x_n)$ and $e_{n+1} = e_n^2/2x_n$.

(b) Show that if $x_0 > \sqrt{c}$, then $x_n > \sqrt{c}$ for all n. Explain graphically.

(c) Show that if $x_0 > \sqrt{c}$, then $e_{n+1} \le e_n^2/(2\sqrt{c})$.

*In Exercises 39–41, a flexible chain of length L is suspended between two poles of equal height separated by a distance $2M$ (Figure 13). By Newton's laws, the chain describes a **catenary** $y = a\cosh(\frac{x}{a})$ where a is the number such that $L = 2a\sinh(\frac{M}{a})$. The sag s is the vertical distance from the highest to the lowest point on the chain.*

39. Suppose that $L = 120$ and $M = 50$.

(a) Use Newton's Method to find a value of a (to two decimal places) satisfying $L = 2a\sinh(M/a)$.

(b) Compute the sag s.

40. Assume that M is fixed.

(a) Calculate $\frac{ds}{da}$. Note that $s = a\cosh\left(\frac{M}{a}\right) - a$.

(b) Calculate $\frac{da}{dL}$ by implicit differentiation using the relation $L = 2a\sinh\left(\frac{M}{a}\right)$.

(c) Use (a) and (b) and the Chain Rule to show that

$$\frac{ds}{dL} = \frac{ds}{da}\frac{da}{dL} = \frac{\cosh(M/a) - (M/a)\sinh(M/a) - 1}{2\sinh(M/a) - (2M/a)\cosh(M/a)} \qquad \boxed{3}$$

41. Suppose that $L = 160$ and $M = 50$.

(a) Use Newton's Method to find a value of a (to two decimal places) satisfying $L = 2a\sinh(M/a)$.

(b) Use Eq. (3) and the Linear Approximation to estimate the increase in sag Δs for changes in length $\Delta L = 1$ and $\Delta L = 5$.

(c) CAS Compute $s(161) - s(160)$ and $s(165) - s(160)$ directly and compare with your estimates in (b).

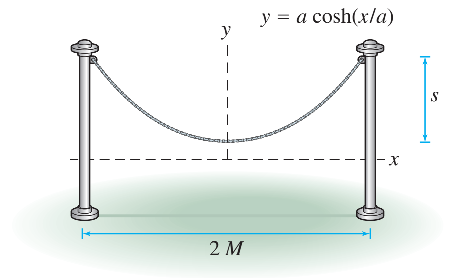


FIGURE 13 Chain hanging between two poles.

CHAPTER REVIEW EXERCISES

In Exercises 1–6, estimate using the Linear Approximation or linearization, and use a calculator to estimate the error.

1. $8.1^{1/3} - 2$

2. $\dfrac{1}{\sqrt{4.1}} - \dfrac{1}{2}$

3. $625^{1/4} - 624^{1/4}$

4. $\sqrt{101}$

5. $\dfrac{1}{1.02}$

6. $\sqrt[5]{33}$

In Exercises 7–12, find the linearization at the point indicated.

7. $y = \sqrt{x}, \quad a = 25$

8. $v(t) = 32t - 4t^2, \quad a = 2$

9. $A(r) = \frac{4}{3}\pi r^3, \quad a = 3$

10. $V(h) = 4h(2-h)(4-2h), \quad a = 1$

11. $P(x) = e^{-x^2/2}, \quad a = 1$

12. $f(x) = \ln(x+e), \quad a = e$

In Exercises 13–16, use the Linear Approximation.

13. The position of an object in linear motion at time t is $s(t) = 0.4t^2 + (t+1)^{-1}$. Estimate the distance traveled over the time interval $[4, 4.2]$.

14. A bond that pays \$10,000 in 6 years is offered for sale at a price P. The percentage yield Y of the bond is

$$Y = 100\left(\left(\frac{10{,}000}{P}\right)^{1/6} - 1\right)$$

Verify that if $P = \$7500$, then $Y = 4.91\%$. Estimate the drop in yield if the price rises to \$7700.

15. When a bus pass from Albuquerque to Los Alamos is priced at p dollars, a bus company takes in a monthly revenue of $R(p) = 1.5p - 0.01p^2$ (in thousands of dollars).

(a) Estimate ΔR if the price rises from \$50 to \$53.

(b) If $p = 80$, how will revenue be affected by a small increase in price? Explain using the Linear Approximation.

16. Show that $\sqrt{a^2+b} \approx a + \frac{b}{2a}$ if b is small. Use this to estimate $\sqrt{26}$ and find the error using a calculator.

17. Use the Intermediate Value Theorem to show that $\sin x - \cos x = 3x$ has a solution, and use Rolle's Theorem to show that this solution is unique.

18. Show that $f(x) = 2x^3 + 2x + \sin x + 1$ has precisely one real root.

19. Verify the MVT for $f(x) = \ln x$ on $[1, 4]$.

20. Suppose that $f(1) = 5$ and $f'(x) \ge 2$ for $x \ge 1$. Use the MVT to show that $f(8) \ge 19$.

21. Use the MVT to prove that if $f'(x) \le 2$ for $x > 0$ and $f(0) = 4$, then $f(x) \le 2x + 4$ for all $x \ge 0$.

22. A function f has derivative $f'(x) = \dfrac{1}{x^4+1}$. Where on the interval $[1, 4]$ does f take on its maximum value?

In Exercises 23–28, find the critical points and determine whether they are minima, maxima, or neither.

23. $f(x) = x^3 - 4x^2 + 4x$

24. $s(t) = t^4 - 8t^2$

25. $f(x) = x^2(x+2)^3$

26. $f(x) = x^{2/3}(1-x)$

27. $g(\theta) = \sin^2\theta + \theta$

28. $h(\theta) = 2\cos 2\theta + \cos 4\theta$

In Exercises 29–36, find the extreme values on the interval.

29. $f(x) = x(10-x), \quad [-1, 3]$

30. $f(x) = 6x^4 - 4x^6, \quad [-2, 2]$

31. $g(\theta) = \sin^2\theta - \cos\theta, \quad [0, 2\pi]$

32. $R(t) = \dfrac{t}{t^2+t+1}, \quad [0, 3]$

33. $f(x) = x^{2/3} - 2x^{1/3}, \quad [-1, 3]$

34. $f(x) = 4x - \tan^2 x, \quad [-\frac{\pi}{4}, \frac{\pi}{3}]$

35. $f(x) = x - 12 \ln x, \quad [5, 40]$

36. $f(x) = e^x - 20x - 1, \quad [0, 5]$

37. Find the critical points and extreme values of $f(x) = |x - 1| + |2x - 6|$ in $[0, 8]$.

38. Match the description of f with the graph of its *derivative* f' in Figure 1.

(a) f is increasing and concave up.

(b) f is decreasing and concave up.

(c) f is increasing and concave down.

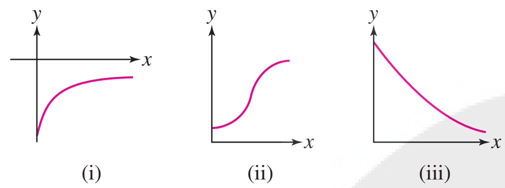


FIGURE 1 Graphs of the derivative.

In Exercises 39–44, find the points of inflection.

39. $y = x^3 - 4x^2 + 4x$

40. $y = x - 2\cos x$

41. $y = \dfrac{x^2}{x^2 + 4}$

42. $y = \dfrac{x}{(x^2 - 4)^{1/3}}$

43. $f(x) = (x^2 - x)e^{-x}$

44. $f(x) = x(\ln x)^2$

In Exercises 45–54, sketch the graph, noting the transition points and asymptotic behavior.

45. $y = 12x - 3x^2$

46. $y = 8x^2 - x^4$

47. $y = x^3 - 2x^2 + 3$

48. $y = 4x - x^{3/2}$

49. $y = \dfrac{x}{x^3 + 1}$

50. $y = \dfrac{x}{(x^2 - 4)^{2/3}}$

51. $y = \dfrac{1}{|x + 2| + 1}$

52. $y = \sqrt{2 - x^3}$

53. $y = \sqrt{3} \sin x - \cos x$ on $[0, 2\pi]$

54. $y = 2x - \tan x$ on $[0, 2\pi]$

55. Draw a curve $y = f(x)$ for which f' and f'' have signs as indicated in Figure 2.

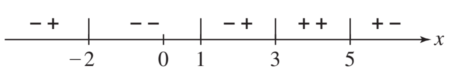


FIGURE 2

56. Find the dimensions of a cylindrical can with a bottom but no top of volume 4 m^3 that uses the least amount of metal.

57. A rectangular open-topped box of height h with a square base of side b has volume $V = 4$ m^3. Two of the side faces are made of material costing \$40/m^2. The remaining sides cost \$20/m^2. Which values of b and h minimize the cost of the box?

58. The corn yield on a certain farm is

$$Y = -0.118x^2 + 8.5x + 12.9 \quad \text{(bushels per acre)}$$

where x is the number of corn plants per acre (in thousands). Assume that corn seed costs \$1.25 (per thousand seeds) and that corn can be sold for \$1.50/bushel. Let $P(x)$ be the profit (revenue minus the cost of seeds) at planting level x.

(a) Compute $P(x_0)$ for the value x_0 that maximizes yield Y.

(b) Find the maximum value of $P(x)$. Does maximum yield lead to maximum profit?

59. Let $N(t)$ be the size of a tumor (in units of 10^6 cells) at time t (in days). According to the **Gompertz Model**, $dN/dt = N(a - b \ln N)$, where a, b are positive constants. Show that the maximum value of N is $e^{a/b}$ and that the tumor increases most rapidly when $N = e^{a/b - 1}$.

60. A truck gets 10 miles per gallon (mpg) of diesel fuel traveling along an interstate highway at 50 mph. This mileage decreases by 0.15 mpg for each mile per hour increase above 50 mph.

(a) If the truck driver is paid \$30/h and diesel fuel costs $P =$ \$3/gal, which speed v between 50 and 70 mph will minimize the cost of a trip along the highway? Notice that the actual cost depends on the length of the trip, but the optimal speed does not.

(b) GU Plot cost as a function of v (choose the length arbitrarily) and verify your answer to part (a).

(c) GU Do you expect the optimal speed v to increase or decrease if fuel costs go down to $P =$ \$2/gal? Plot the graphs of cost as a function of v for $P = 2$ and $P = 3$ on the same axis and verify your conclusion.

61. Find the maximum volume of a right-circular cone placed upside-down in a right-circular cone of radius $R = 3$ and height $H = 4$ as in Figure 3. A cone of radius r and height h has volume $\frac{1}{3}\pi r^2 h$.

62. Redo Exercise 61 for arbitrary R and H.

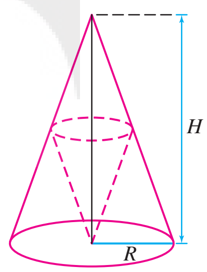


FIGURE 3

63. Show that the maximum area of a parallelogram $ADEF$ that is inscribed in a triangle ABC, as in Figure 4, is equal to one-half the area of $\triangle ABC$.

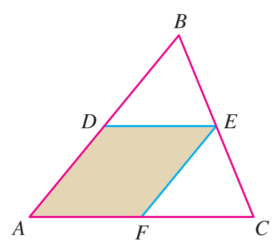


FIGURE 4

64. A box of volume 8 m^3 with a square top and bottom is constructed out of two types of metal. The metal for the top and bottom costs \$50/m^2 and the metal for the sides costs \$30/m^2. Find the dimensions of the box that minimize total cost.

65. Let f be a function whose graph does not pass through the x-axis and let $Q = (a, 0)$. Let $P = (x_0, f(x_0))$ be the point on the graph closest to Q (Figure 5). Prove that $\overline{PQ}$ is perpendicular to the tangent line to the graph of x_0. *Hint:* Find the minimum value of the *square* of the distance from $(x, f(x))$ to $(a, 0)$.

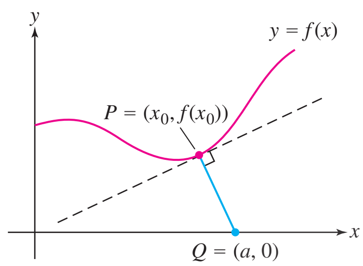


FIGURE 5

66. Take a circular piece of paper of radius R, remove a sector of angle θ (Figure 6), and fold the remaining piece into a cone-shaped cup. Which angle θ produces the cup of largest volume?

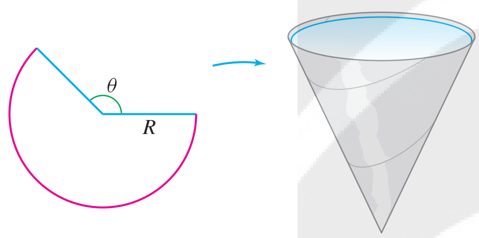


FIGURE 6

67. Use Newton's Method to estimate $\sqrt[3]{25}$ to four decimal places.

68. Use Newton's Method to find a root of $f(x) = x^2 - x - 1$ to four decimal places.

69. Find the local extrema of $f(x) = \dfrac{e^{2x} + 1}{e^{x+1}}$.

70. Find the points of inflection of $f(x) = \ln(x^2 + 1)$ and, at each point, determine whether the concavity changes from up to down or from down to up.

In Exercises 71–74, find the local extrema and points of inflection, and sketch the graph. Use L'Hôpital's Rule to determine the limits as $x \to 0+$ or $x \to \pm\infty$ if necessary.

71. $y = x \ln x \quad (x > 0)$

72. $y = e^{x - x^2}$

73. $y = x(\ln x)^2 \quad (x > 0)$

74. $y = \tan^{-1}\left(\dfrac{x^2}{4}\right)$

75. Explain why L'Hôpital's Rule gives no information about $\displaystyle\lim_{x\to\infty} \frac{2x - \sin x}{3x + \cos 2x}$. Evaluate the limit by another method.

76. Let f be a differentiable function with inverse g which is also differentiable. Assume that $f(0) = 0$ and $f'(0) \neq 0$.

(a) Use the fact that $f(g(x)) = x$ and the Chain Rule to show that $g'(x) = \dfrac{1}{f'(g(x))}$.

(b) Prove that

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = f'(0)^2$$

In Exercises 77–88, verify that L'Hôpital's Rule applies and evaluate the limit.

77. $\displaystyle\lim_{x\to 3} \frac{4x - 12}{x^2 - 5x + 6}$

78. $\displaystyle\lim_{x\to -2} \frac{x^3 + 2x^2 - x - 2}{x^4 + 2x^3 - 4x - 8}$

79. $\displaystyle\lim_{x\to 0+} x^{1/2} \ln x$

80. $\displaystyle\lim_{t\to\infty} \frac{\ln(e^t + 1)}{t}$

81. $\displaystyle\lim_{\theta\to 0} \frac{2\sin\theta - \sin 2\theta}{\sin\theta - \theta\cos\theta}$

82. $\displaystyle\lim_{x\to 0} \frac{\sqrt{4+x} - 2\sqrt[8]{1+x}}{x^2}$

83. $\displaystyle\lim_{t\to\infty} \frac{\ln(t+2)}{\log_2 t}$

84. $\displaystyle\lim_{x\to 0} \left(\frac{e^x}{e^x - 1} - \frac{1}{x}\right)$

85. $\displaystyle\lim_{y\to 0} \frac{\sin^{-1} y - y}{y^3}$

86. $\displaystyle\lim_{x\to 1} \frac{\sqrt{1 - x^2}}{\cos^{-1} x}$

87. $\displaystyle\lim_{x\to 0} \frac{\sinh(x^2)}{\cosh x - 1}$

88. $\displaystyle\lim_{x\to 0} \frac{\tanh x - \sinh x}{\sin x - x}$

89. Let $f(x) = e^{-Ax^2/2}$, where $A > 0$ is a constant. Given any n numbers $a_1, a_2, \dots, a_n$, set

$$\Phi(x) = f(x - a_1) f(x - a_2) \cdots f(x - a_n)$$

(a) Assume $n = 2$ and prove that Φ attains its maximum value at the average $x = \frac{1}{2}(a_1 + a_2)$. *Hint:* Calculate $\Phi'(x)$ using logarithmic differentiation.

(b) Show that for any n, Φ attains its maximum value at

$$x = \frac{1}{n}(a_1 + a_2 + \cdots + a_n).$$

This fact is related to the role of $f(x)$ (whose graph is a bell-shaped curve) in statistics.